

Schatten class and Berezin transform of quaternionic linear operators

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Abstract

In this paper we introduce the Schatten class of operators and the Berezin transform of operators in the quaternionic setting. The first topic is of great importance in operator theory but it is also necessary to study the second one because we need the notion of trace class operators, which is a particular case of the Schatten class. Regarding the Berezin transform, we give the general definition and properties. Then we concentrate on the setting of weighted Bergman spaces of slice hyperholomorphic functions. Our results are based on the S -spectrum for quaternionic operators, which is the notion of spectrum that appears in the quaternionic version of the spectral theorem and in the quaternionic S -functional calculus.

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1 Introduction

The study of quaternionic linear operators was stimulated by the celebrated paper of G. Birkhoff and J. von Neumann [13] on the logic of quantum mechanics, where they proved that there are essentially two possible ways to formulate quantum mechanics: the well known one using complex numbers and also one using quaternions. The development of quaternionic quantum mechanics has been done by several authors in the past. Without claiming completeness, we mention: S. Adler, see [1], L. P. Horwitz and L. C. Biedenharn, see [31], D. Finkelstein, J. M. Jauch, S. Schiminovich and D. Speiser, see [26], and G. Emch see [23]. The notion of spectrum of a quaternionic linear operator used in the past was the right spectrum. Even though

this notion of spectrum gives just a partial description of the spectral analysis of a quaternionic operator, basically it contains just the eigenvalues, several results in quaternionic quantum mechanics have been obtained.

Only in 2007, one of the authors introduced with some collaborators the notion of S -spectrum of a quaternionic operator, see the book [20]. This spectrum was suggested by the quaternionic version of the Riesz-Dunford functional calculus (see [21,32] for the classical results), which is called S -functional calculus or quaternionic functional calculus and whose full development was done in [12,18,19]. Using the notion of right spectrum of a quaternionic matrix, the spectral theorem was proved in [24] (see [22] for the classical result). The spectral theorem for quaternionic bounded or unbounded normal operators based on the S -spectrum has been proved recently in the papers [3,5]. The case of compact operators is in [29].

The literature contains several attempts to prove the quaternionic version of the spectral theorem, but the notion of spectrum in the general case was not made clear, see [34,37]. The fact that the S -eigenvalues and the right eigenvalues are equal explains why some results in quaternionic quantum mechanic were possible to be obtained without the notion of S -spectrum. More recently, the use of the S -spectrum can be found in [35].

The aim of this paper is to introduce the Schatten class of quaternionic operators and the Berezin transform. In the following, a complex Hilbert space will be denoted by $\mathcal{H}_{\mathbb{C}}$ while we keep the symbol \mathcal{H} for a quaternionic Hilbert space. We recall some facts of the classical theory in order to better explain the quaternionic setting. Suppose that B is a compact self-adjoint operator on a complex separable Hilbert space $\mathcal{H}_{\mathbb{C}}$. Then it is well known that there exists a sequence of real numbers $\{\lambda_n\}$ tending to zero and there exists an orthonormal set $\{u_n\}$ in $\mathcal{H}_{\mathbb{C}}$ such that

$$Bx = \sum_{n=1}^{\infty} \lambda_n \langle x, u_n \rangle u_n, \quad \text{for all } x \in \mathcal{H}_{\mathbb{C}}.$$

When A is a compact operator, but not necessarily self-adjoint, we consider the polar decomposition $A = U|A|$, where $|A| = \sqrt{A^*A}$ is a positive operator and U is a partial isometry. From the spectral representation of compact self-adjoint operators and the polar decomposition of bounded linear operators, we get the representation of the compact operator A as

$$Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, u_n \rangle v_n, \quad \text{for all } x \in \mathcal{H}_{\mathbb{C}},$$

where $v_n := Uu_n$ is an orthonormal set. The positive real numbers $\{\lambda_n\}_{n \in \mathbb{N}}$ are called singular values of A .

Associated with the singular values of A , we have the Schatten class operators S_p that consists of those operators for which the sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ belongs to ℓ^p for $p \in (0, \infty)$. In the case $p \in [1, \infty)$ the sets S_p are Banach spaces.

Two important spaces are $p = 1$, the so called trace class, and $p = 2$, the Hilbert-Smith operators.

The Schatten class operators were apparently introduced to study integral equations with non symmetric kernels as pointed out in the book of I. C. Gohberg and M. G. Krein [30], later on it was discovered that the properties of the singular values play an important role in the construction of the general theory of symmetrically normed ideals of compact operators.

Let us point out that in the quaternionic setting, it has been understood just recently what the correct notion of spectrum is that replaces the classical notion of spectrum of operators on a complex Banach space. This spectrum is called S -spectrum and its is also defined for n -tuples of noncommuting operators, see [20].

Precisely, let V be a bilateral quaternionic Banach space and denote by $\mathcal{B}(V)$ the Banach space of all quaternionic bounded linear operators on V endowed with the natural norm. Let $T : V \rightarrow V$ be a quaternionic bounded linear operator (left or right linear). We define the S -spectrum of T as

$$\sigma_S(T) = \{s \in \mathbb{H} : T^2 - 2\text{Re}(s)T + |s|^2\mathcal{I} \text{ is not invertible in } \mathcal{B}(V)\}$$

where $s = s_0 + s_1e_1 + s_2e_2 + s_3e_3$ is a quaternion and e_1, e_2 and e_3 is the standard basis of the quaternions \mathbb{H} , $\text{Re}(s) = s_0$ is the real part and $|s|^2$ is the squared Euclidean norm. The S -resolvent set is defined as

$$\rho_S(T) = \mathbb{H} \setminus \sigma_S(T).$$

The notion of S -spectrum for quaternionic operators arises naturally in the slice hyperholomorphic functional calculus, called S -functional calculus or quaternionic functional calculus, which is the quaternionic analogue of the Riesz-Dunford functional calculus for complex operators on a complex Banach space. Recently, it turned out that also the spectral theorem for quaternionic operators (bounded or unbounded) is based on the S -spectrum.

This fact restores for the quaternionic setting the well known fact that both the Riesz-Dunford functional calculus and the spectral theorem are based on the same notion of spectrum.

This was not clear for a long time because the literature related to quaternionic linear operators used the notion of right spectrum of a quaternionic operator, which, for a right linear operator T , is defined as

$$\sigma_R(T) = \{s \in \mathbb{H} \text{ such that } \exists 0 \neq v \in V : Tv = vs\}.$$

This definition gives only rise to the eigenvalues of T . The other possibility is to define the left spectrum $\sigma_L(T)$, where we replace $Tv = vs$ by $Tv = sv$. In the case of the right eigenvalues we have a nonlinear operator associated with the spectral problem. In the case of the left spectrum, we have a linear equation, but in both cases there does not exist a suitable notion of resolvent operator which preserves some sort of hyperholomorphicity.

Only in the case of the S -spectrum, we can associate to it the two S -resolvent operators. For $s \in \rho_S(T)$, the left S -resolvent operator

$$S_L^{-1}(s, T) := -(T^2 - 2\text{Re}(s)T + |s|^2\mathcal{I})^{-1}(T - \overline{s}\mathcal{I}),$$

and the right S -resolvent operator

$$S_R^{-1}(s, T) := -(T - \overline{s}\mathcal{I})(T^2 - 2\text{Re}(s)T + |s|^2\mathcal{I})^{-1}.$$

are operator-valued slice hyperholomorphic functions. Moreover, with the notion of S -spectrum, the usual decompositions of the spectrum turn out to be natural, for example in point spectrum, continuous spectrum and residual spectrum.

So one of the main aims of this paper is to define the Schatten classes of quaternionic operators using the notion of S -spectrum and to study their main properties.

Here we point out the following fact that shows one of the main differences with respect to the complex case. The trace of a compact complex linear operator is defined as

$$\text{tr } A = \sum_{n \in \mathbb{N}} \langle e_n, Ae_n \rangle, \tag{1}$$

where $(e_n)_{n \in \mathbb{N}}$ is an orthonormal basis of the Hilbert space $\mathcal{H}_{\mathbb{C}}$. It can be shown that the trace is independent of the choice of the orthonormal basis and therefore well-defined. If we try to generalize the above definition to the quaternionic setting, we face the problem that this invariance with respect to the choice of the orthonormal basis does not persist. We have to define the trace in a different way. In fact we have to restrict to operators that satisfy an additional condition: *Let $J \in \mathcal{B}(\mathcal{H})$ be an anti-selfadjoint and unitary operator. For $p \in (0, +\infty]$, we define the (J, p) -Schatten class of operators $S_p(J)$ as*

$$S_p(J) := \{T \in \mathcal{B}_0(\mathcal{H}) : [T, J] = 0 \text{ and } (\lambda_n(T))_{n \in \mathbb{N}} \in \ell^p\},$$

where $(\lambda_n(T))_{n \in \mathbb{N}}$ denotes the sequence of singular values of T .

The Berezin transform is a useful tool to study operators acting on spaces of holomorphic functions such as

Bergman spaces. In this paper we treat the case of quaternionic operators acting on spaces of slice hyperholomorphic functions.

We recall some facts of the classical theory, see [33] for more details. To state the definition of the Berezin transform of a bounded operator we recall that:

A reproducing kernel Hilbert space on an open set Ω of \mathbb{C} , is a Hilbert space $\mathcal{H}_{\mathbb{C}}$ of functions on Ω such that for every $w \in \Omega$ the linear functional $f \rightarrow f(w)$ is bounded on $\mathcal{H}_{\mathbb{C}}$.

When $\mathcal{H}_{\mathbb{C}}$ is a reproducing kernel Hilbert space, by the Riesz representation theorem, there exists a unique $K_w \in \mathcal{H}_{\mathbb{C}}$ for every $w \in \Omega$ such that $f(w) = \langle f, K_w \rangle$ for all $f \in \mathcal{H}_{\mathbb{C}}$. The function K_w is called a reproducing kernel. It is a well known fact that $K_w(z) = \sum_{\ell \in L} \overline{e_{\ell}(w)} e_{\ell}(z)$ where $\{e_{\ell} : \ell \in L\}$ is an orthonormal basis for $\mathcal{H}_{\mathbb{C}}$. Since $K_w(z) = \overline{K_z(w)}$, we write $K(z, w) := K_w(z)$. The norm of K_w is given by

$$\|K_w\|^2 = \langle K_w, K_w \rangle = K_w(w) = K(w, w)$$

and when this norm is different from zero the function

$$k_w = \frac{K_w}{\|K_w\|}$$

is called normalized reproducing kernel at w . Examples of reproducing kernel Hilbert spaces are Hardy spaces, Bergman spaces, Fock spaces and Dirichlet spaces to name a few.

Keeping in mind the above definition, we define the Berezin transform of an operator as follows: *Let $\mathcal{H}_{\mathbb{C}}$ be a reproducing kernel Hilbert space of analytic functions on an open set Ω of \mathbb{C} and let A be a bounded operator on $\mathcal{H}_{\mathbb{C}}$. The Berezin transform of A is defined as*

$$\tilde{A}(w) := \langle K_w, AK_w \rangle, \quad \text{for all } w \in \Omega.$$

The importance of the Berezin transform is due to the possibility to deduce properties of the bounded operator A from the properties of the analytic function $\tilde{A}(w)$. For example, an important consequence is that

$$A = 0 \quad \text{if and only if} \quad \tilde{A}(w) = 0 \quad \text{for all } w \in \Omega.$$

When we consider quaternionic operators, we have to replace the notion of holomorphic function with the notion of slice hyperholomorphic function. Using the theory of function spaces of slice hyperholomorphic functions, it is possible to extend classical results to the quaternionic setting.

We treat the case of weighted Bergman spaces on the unit ball \mathbb{D} in the quaternions, whose reproducing kernel is defined as follows: consider $\alpha > -1$ and $q, w \in \mathbb{D}$. For $q = q_0 + i_q q_1$ set $q_{i_w} = q_0 + i_w q_1$, where $w = w_0 + i_w w_1$, and define the slice hyperholomorphic α -Bergman kernel as

$$K_{\alpha}(q, w) := \frac{1}{2}(1 - i_q i) \frac{1}{(1 - q_{i_w} \overline{w})^{2+\alpha}} + \frac{1}{2}(1 + i_q i) \frac{1}{(1 - \overline{q_{i_w}} w)^{2+\alpha}}.$$

This kernel is left slice hyperholomorphic in q and anti right slice hyperholomorphic in w . Moreover, whenever q and w belong to the same complex plane, it reduces to the complex Bergman kernel.

The plan of the paper is as follows:

Section 1 contains the introduction.

Section 2 contains a subsection in which we discuss two of the possible definitions of slice hyperholomorphic functions and their properties, and a subsection that contains the main results on quaternionic operator theory.

Section 3 contains the Schatten class of quaternionic operators. Precisely, we consider the singular values of compact operators, the Schatten class, the dual of the Schatten class and different characterizations of Schatten class operators.

Section 4 contains the general theory of the Berezin transform and the case of weighted Bergman spaces with several properties.

2 Preliminary results

Recent works on Schur analysis in the slice hyperholomorphic setting introduced and studied the quaternionic Hardy spaces $H_2(\Omega)$, where Ω is the quaternionic unit ball \mathbb{D} or the half space \mathbb{H}^+ of quaternions with positive real part, [7, 8, 11]. The Hardy spaces $H^p(\mathbb{B})$ for $p > 2$ are considered in [36]. The Bergman spaces are treated in [15–17] and for the Fock spaces see [10]. Weighted Bergman spaces, Bloch, Besov and Dirichlet spaces on the unit ball \mathbb{B} are studied in [14]. In the following we want discuss two possible definitions of slice hyperholomorphic functions since in both cases the above mentioned function spaces can be defined with minor changes in the proofs. The discussion in the next subsection will clarify the variations of slice hyperholomorphicity.

2.1 Slice hyperholomorphic functions

The skew-field of quaternions consists of the real vector space

$$\mathbb{H} := \left\{ \xi_0 + \sum_{\ell=1}^3 \xi_\ell e_\ell : \xi_\ell \in \mathbb{R} \right\},$$

where the units e_1, e_2 and e_3 satisfy $e_i^2 = -1$ and $e_\ell e_\kappa = -e_\ell e_\kappa$ for $\ell, \kappa \in \{1, 2, 3\}$ with $\ell \neq \kappa$. The real part of a quaternion $x = \xi_0 + \sum_{\ell=1}^3 \xi_\ell e_\ell$ is defined as $\text{Re}(x) := \xi_0$, its imaginary part as $\underline{x} := \sum_{\ell=1}^3 \xi_\ell e_\ell$ and its conjugate as $\bar{x} := \text{Re}(x) - \underline{x}$. Each element of the set

$$\mathbb{S} := \{x \in \mathbb{H} : \text{Re}(x) = 0, |x| = 1\}$$

is a square-root of -1 and is therefore called an imaginary unit. For any $i \in \mathbb{S}$, the subspace $\mathbb{C}_i := \{x_0 + ix_1 : x_1, x_2 \in \mathbb{R}\}$ is isomorphic to the field of complex numbers. For $i, j \in \mathbb{S}$ with $i \perp j$, set $k = ij = -ji$. Then $1, i, j$ and k form an orthonormal basis of \mathbb{H} as a real vector space and 1 and j form an orthonormal basis of \mathbb{H} as a left or right vector space over the complex plane \mathbb{C}_i , that is

$$\mathbb{H} = \mathbb{C}_i + \mathbb{C}_i j \quad \text{and} \quad \mathbb{H} = \mathbb{C}_i + j\mathbb{C}_i.$$

Any quaternion x belongs to such a complex plane: if we set

$$i_x := \begin{cases} \underline{x}/|\underline{x}|, & \text{if } \underline{x} \neq 0 \\ \text{any } i \in \mathbb{S}, & \text{if } \underline{x} = 0, \end{cases}$$

then $x = x_0 + i_x x_1$ with $x_0 = \text{Re}(x)$ and $x_1 = |\underline{x}|$. The set

$$[x] := \{x_0 + ix_1 : i \in \mathbb{S}\},$$

is a 2-sphere, that reduces to a single point if x is real. We recall a well known result that we will need in the sequel.

Lemma 2.1. *Let $x, y \in \mathbb{H}$. Then $y \in [x]$ if and only if there exists $q \in \mathbb{H}$ with $|q| = 1$ such that $y = q^{-1}xq$.*

The notion of slice hyperholomorphicity is the generalization of holomorphicity to quaternion-valued functions that underlies the theory of quaternionic linear operators. We recall the main results on slice hyperholomorphic functions. The proofs of the results stated in this subsection can be found in the book [20].

Definition 2.2. A set $U \subset \mathbb{H}$ is called

- (i) axially symmetric if $[x] \subset U$ for any $x \in U$ and
- (ii) a slice domain if U is open, $U \cap \mathbb{R} \neq \emptyset$ and $U \cap \mathbb{C}_i$ is a domain for any $i \in \mathbb{S}$.

Definition 2.3. Let $U \subset \mathbb{H}$ be an axially symmetric open set. A real differentiable function $f : U \rightarrow \mathbb{H}$ is called left slice hyperholomorphic if it has the form

$$f(x) = \alpha(x_0, x_1) + i_x \beta(x_0, x_1), \quad \forall x = x_0 + i_x x_1 \in U \quad (2)$$

such that the functions α and β , which take values in \mathbb{H} , satisfy the compatibility condition

$$\begin{aligned} \alpha(x_0, x_1) &= \alpha(x_0, x_1) \\ \beta(x_0, x_1) &= -\beta(x_0, -x_1) \end{aligned} \quad (3)$$

and the Cauchy-Riemann-system

$$\begin{aligned} \frac{\partial}{\partial x_0} \alpha(x_0, x_1) &= \frac{\partial}{\partial x_1} \beta(x_0, x_1) \\ \frac{\partial}{\partial x_0} \beta(x_0, x_1) &= -\frac{\partial}{\partial x_1} \alpha(x_0, x_1). \end{aligned} \quad (4)$$

A function $f : U \rightarrow \mathbb{H}$ is called right slice hyperholomorphic if it has the form

$$f(x) = \alpha(x_0, x_1) + \beta(x_0, x_1) i_x, \quad \forall x = x_0 + i_x x_1 \in U, \quad (5)$$

such that the functions α and β satisfy (3) and (4).

The sets of left and right slice hyperholomorphic functions on U are denoted by $\mathcal{SH}_L(U)$ and $\mathcal{SH}_R(U)$, respectively. Finally, we say that a function f is left or right slice hyperholomorphic on a closed axially symmetric set K , if there exists an open axially symmetric set U with $K \subset U$ such that $f \in \mathcal{SH}_L(U)$ resp. $\mathcal{SH}_R(U)$.

Corollary 2.4. Let $U \subset \mathbb{H}$ be axially symmetric.

- (i) If $f, g \in \mathcal{SH}_L(U)$ and $a \in \mathbb{H}$, then $fa + g \in \mathcal{SH}_L(U)$.
- (ii) If $f, g \in \mathcal{SH}_R(U)$ and $a \in \mathbb{H}$, then $af + g \in \mathcal{SH}_R(U)$.

On axially symmetric slice domains, slice hyperholomorphic functions can be characterized as those functions that lie in the kernel of a slicewise Cauchy-Riemann-operator. As a consequence, the restriction of a slice hyperholomorphic function to a complex plane can be split into two holomorphic components.

Definition 2.5. Let f be a function defined on a set $U \subset \mathbb{H}$ and let $i \in \mathbb{S}$. We denote the restriction of f to the complex plane \mathbb{C}_i by f_i , i.e. $f_i := f|_{U \cap \mathbb{C}_i}$.

In order to avoid confusion we point out that, in this paper, indices i and j always refer to restrictions of a function to the respective complex planes.

Definition 2.6. Let $U \subset \mathbb{H}$ be open. We define the following differential operators: for any real differentiable function $f : U \rightarrow \mathbb{H}$ and any $x = x_0 + i_x x_1 \in U$, we set

$$\begin{aligned} \partial_i f(x) &= \frac{1}{2} \left(\frac{\partial}{\partial x_0} f_{i_x}(x) - i_x \frac{\partial}{\partial x_1} f_{i_x}(x) \right) \\ \bar{\partial}_i f(x) &= \frac{1}{2} \left(\frac{\partial}{\partial x_0} f_{i_x}(x) + i_x \frac{\partial}{\partial x_1} f_{i_x}(x) \right) \end{aligned}$$

and

$$\begin{aligned} f(x) \partial_{\leftarrow i} &= \frac{1}{2} \left(\frac{\partial}{\partial x_0} f_i(x) - \frac{\partial}{\partial x_1} f_i(x) i_x \right) \\ f(x) \bar{\partial}_{\leftarrow i} &= \frac{1}{2} \left(\frac{\partial}{\partial x_0} f_i(x) + \frac{\partial}{\partial x_1} f_i(x) i_x \right), \end{aligned}$$

where the arrow \leftarrow indicates that the operators $\bar{\partial}_{\leftarrow i}$ and $\partial_{\leftarrow i}$ act from the right.

If a function depends on several variables and we want to stress that these operators act in a variable x , then we write ∂_{ix} instead of ∂_i etc.

Corollary 2.7. *Let $U \subset \mathbb{H}$ be open and axially symmetric.*

- (i) *If $f \in \mathcal{SH}_L(U)$, then $\bar{\partial}_i f = 0$. If U is a slice domain, then $f \in \mathcal{SH}_L(U)$ if and only if $\bar{\partial}_i f = 0$.*
- (ii) *If $f \in \mathcal{SH}_R(U)$, then $f \bar{\partial}_{i^c} = 0$. If U is a slice domain, then $f \in \mathcal{SH}_R(U)$ if and only if $f \bar{\partial}_{i^c} = 0$.*

Lemma 2.8 (Splitting Lemma). *Let $U \subset \mathbb{H}$ be axially symmetric and let $i, j \in \mathbb{S}$ with $i \perp j$.*

- (i) *If $f \in \mathcal{SH}_L(U)$, then there exist holomorphic functions $f_1, f_2 : U \cap \mathbb{C}_i \rightarrow \mathbb{C}_i$ such that $f_i = f_1 + f_2 j$.*
- (ii) *If $f \in \mathcal{SH}_R(U)$, then there exist holomorphic functions $f_1, f_2 : U \cap \mathbb{C}_i \rightarrow \mathbb{C}_i$ such that $f_i = f_1 + j f_2$.*

Remark 2.9. Originally, in particular in [20], slice hyperholomorphic functions were defined as functions that satisfy $\bar{\partial}_i f = 0$ resp. $f \bar{\partial}_{i^c} = 0$. In principle, this leads to a larger class of functions, but on axially symmetric slice domains both definitions are equivalent. Indeed, on an axially symmetric slice domain, $\bar{\partial}_i f = 0$ resp. $f \bar{\partial}_{i^c} = 0$ implies that f satisfies the representation formula (cf. Theorem 2.15), which allows for a representation of f of the form (2) resp. (5).

The theory of slice hyperholomorphicity was therefore only developed for functions that are defined on axially symmetric slice domains. However, most results do not directly depend on the fact that the functions are defined on a slice domain, but rather on their representation of the form (2) resp. (5). Hence, the definition given in this paper seems to be more appropriate since it allows to extend the theory also to functions defined on sets that are not connected or do not intersect the real line.

Definition 2.10. Let $U \subset \mathbb{H}$ be axially symmetric. A left slice hyperholomorphic $f(x) = \alpha(x_0, x_1) + i_x \beta(x_0, x_1)$ is called intrinsic if α and β are real-valued. We denote the set of intrinsic functions on U by $\mathcal{N}(U)$.

Note that an intrinsic function is both left and right slice hyperholomorphic because $\beta(x_0, x_1)$ commutes with the imaginary unit i_x . The converse is not true: the constant function $x \mapsto b \in \mathbb{H} \setminus \mathbb{R}$ is left and right slice hyperholomorphic, but it is not intrinsic.

The importance of the class of intrinsic functions is due to the fact that the multiplication and composition with intrinsic functions preserve slice hyperholomorphicity. This is not true for arbitrary slice hyperholomorphic functions.

Corollary 2.11. *Let $U \subset \mathbb{H}$ be axially symmetric.*

- (i) *If $f \in \mathcal{N}(U)$ and $g \in \mathcal{SH}_L(U)$, then $fg \in \mathcal{SH}_L(U)$. If $f \in \mathcal{SH}_R(U)$ and $g \in \mathcal{N}(U)$, then $fg \in \mathcal{SH}_R(U)$.*
- (ii) *If $g \in \mathcal{N}(U)$ and $f \in \mathcal{SH}_L(g(U))$, then $f \circ g \in \mathcal{SH}_L(U)$. If $g \in \mathcal{N}(U)$ and $f \in \mathcal{SH}_R(g(U))$, then $f \circ g \in \mathcal{SH}_R(U)$.*

Important examples of slice hyperholomorphic functions are power series with quaternionic coefficients: series of the form $\sum_{n=0}^{+\infty} x^n a_n$ are left slice hyperholomorphic and series of the form $\sum_{n=0}^{\infty} a_n x^n$ are right slice hyperholomorphic on their domain of convergence. A power series is intrinsic if and only if its coefficients are real.

Conversely, any slice hyperholomorphic function can be expanded into a power series at any real point.

Definition 2.12. The slice-derivative of a function $f \in \mathcal{SH}_L(U)$ is defined as

$$\partial_S f(x) = \lim_{\mathbb{C}_{i_x} \ni s \rightarrow x} (s - x)^{-1} (f(s) - f(x)),$$

where $\lim_{\mathbb{C}_{i_x} \ni s \rightarrow x} g(s)$ is the limit as s tends to $x = x_0 + i_x x_1 \in U$ in \mathbb{C}_{i_x} . The slice-derivative of a function $f \in \mathcal{SH}_R(U)$ is defined as

$$f \partial_{i^c} (x) = \lim_{\mathbb{C}_{i_x} \ni s \rightarrow x} (f(s) - f(x))(s - x)^{-1}.$$

If a function depends on several variables and we want to stress that we take the slice-derivative in the variable x , then we write ∂_{Sx} and ∂_{Sx}^\leftarrow instead of ∂_S and ∂_S^\leftarrow .

Corollary 2.13. *The slice derivative of a left (or right) slice hyperholomorphic function is again left (or right) slice hyperholomorphic. Moreover, it coincides with the derivative with respect to the real part, that is*

$$\partial_S f(x) = \frac{\partial}{\partial x_0} f(x) \quad \text{resp.} \quad f \partial_S^\leftarrow(x) = \frac{\partial}{\partial x_0} f(x).$$

Theorem 2.14. *If f is left slice hyperholomorphic on the ball $B_r(\alpha)$ with radius r centered at $\alpha \in \mathbb{R}$, then*

$$f(x) = \sum_{n=0}^{+\infty} (x - \alpha)^n \frac{1}{n!} \partial_S^n f(\alpha) \quad \text{for } x \in B_r(\alpha).$$

If f is right slice hyperholomorphic on $B(r, \alpha)$, then

$$f(x) = \sum_{n=0}^{+\infty} \frac{1}{n!} f \partial_S^n(\alpha) (x - \alpha)^n \quad \text{for } x \in B_r(\alpha).$$

As a consequence of the slice structure of slice hyperholomorphic functions, their values are uniquely determined by their values on an arbitrary complex plane. Consequently, any function that is holomorphic on a suitable subset of a complex plane possesses an unique slice hyperholomorphic extension.

Theorem 2.15 (Representation Formula). *Let $U \subset \mathbb{H}$ be axially symmetric and let $i \in \mathbb{S}$. For any $x = x_0 + i_x x_1 \in U$ set $x_i := x_0 + i x_1$. If $f \in \mathcal{SH}_L(U)$. Then*

$$f(x) = \frac{1}{2}(1 - i_x i) f(x_i) + \frac{1}{2}(1 + i_x i) f(\overline{x_i}).$$

If $f \in \mathcal{SH}_R(U)$, then

$$f(x) = f(x_i)(1 - i i_x) \frac{1}{2} + f(\overline{x_i})(1 + i i_x) \frac{1}{2}.$$

Corollary 2.16. *Let $i \in \mathbb{S}$ and let $f : O \rightarrow \mathbb{H}$ be real differentiable, where O is a domain in \mathbb{C}_i that is symmetric with respect to the real axis.*

- (i) *The axially symmetric hull $[O] := \bigcup_{z \in O} [z]$ of O is an axially symmetric slice domain.*
- (ii) *If f satisfies $\frac{\partial}{\partial x_0} f + i \frac{\partial}{\partial x_1} f = 0$, then there exists a unique left slice hyperholomorphic extension $\text{ext}_L(f)$ of f to $[O]$.*
- (iii) *If f satisfies $\frac{\partial}{\partial x_0} f + \frac{\partial}{\partial x_1} f i = 0$, then there exists a unique right slice hyperholomorphic extension $\text{ext}_R(f)$ of f to $[O]$.*

Remark 2.17. If f has a left and a right slice hyperholomorphic extension, they do not necessarily coincide. Consider for instance the function $z \mapsto bz$ on \mathbb{C}_i with a constant $b \in \mathbb{C}_i \setminus \mathbb{R}$. Its left slice hyperholomorphic extension to \mathbb{H} is $x \mapsto xb$, but its right slice hyperholomorphic extension is $x \mapsto bx$.

2.2 Quaternionic linear operators

In this section, we consider bounded linear operators on a separable quaternionic Hilbert space, even though some definitions and results hold also for quaternionic Banach spaces.

Definition 2.18. Let \mathcal{H} be a quaternionic right vector space together with a scalar product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{H}$ with the following properties:

- (i) positivity: $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$
- (ii) right-linearity: $\langle x, ya + z \rangle = \langle x, y \rangle a + \langle x, z \rangle$ for all $x, y, z \in \mathcal{H}$ and all $a \in \mathbb{H}$.
- (iii) quaternionic hermiticity: $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in \mathcal{H}$.

If \mathcal{H} is complete with respect to the norm

$$\|x\|_{\mathcal{H}} := \sqrt{\langle x, x \rangle},$$

then $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is called a quaternionic Hilbert space.

Terms such as orthogonality, orthonormal basis etc. are defined as in the complex case. In the following, we shall always assume that \mathcal{H} is separable.

We consider now bounded quaternionic right linear operators on \mathcal{H} .

Definition 2.19. A quaternionic right linear operator $T : \mathcal{H} \rightarrow \mathcal{H}$ is called bounded if $\|T\| := \sup_{\|x\|=1} \|Tx\| < +\infty$. We denote the set of all bounded quaternionic right linear operators on \mathcal{H} by $\mathcal{B}(\mathcal{H})$.

The space $\mathcal{B}(\mathcal{H})$ together with the operator norm is a real Banach space. Observe that there does not exist any natural multiplication of operators with scalars in $\mathbb{H} \setminus \mathbb{R}$ because there is no left-multiplication defined on \mathcal{H} . Hence, the operator Ts , which is supposed to act as $Ts(v) = T(sv)$, has no meaning.

Let $T \in \mathcal{B}(\mathcal{H})$. For $s \in \mathbb{H}$, we set

$$Q_s(T) := T^2 - 2\operatorname{Re}(s)T + \mathcal{I}|s|^2,$$

where \mathcal{I} denotes the identity operator.

Definition 2.20. We define the S -resolvent set of an operator $T \in \mathcal{B}(\mathcal{H})$ as

$$\rho_S(T) := \{s \in \mathbb{H} : Q_s(T)^{-1} \in \mathcal{B}(\mathcal{H})\}$$

and the S -spectrum of T as

$$\sigma_S(T) := \mathbb{H} \setminus \rho_S(T).$$

Remark 2.21. For operators acting on a two-sided quaternionic Banach space, the left and right S -resolvent operators are defined by formally replacing the variable x in the left and right Cauchy kernel by the operator T . This motivates the definition of the S -resolvent set and the S -spectrum as it is done in Definition 2.20, cf. [20].

Theorem 2.22 (See [20]). *Let $T \in \mathcal{B}(\mathcal{H})$. Then $\sigma_S(T)$ is an axially symmetric, compact and nonempty subset of $\overline{B_{\|T\|}(0)}$.*

In [28], the following natural partition of the S -spectrum was introduced:

1. the spherical point spectrum of T :

$$\sigma_{pS}(T) := \{s \in \mathbb{H} : \ker(Q_s(T)) \neq \{0\}\}.$$

2. the spherical residual spectrum of T :

$$\sigma_{rS}(T) := \{s \in \mathbb{H} : \ker(Q_s(T)) = \{0\}, \overline{\operatorname{ran}(Q_s(T))} \neq \mathcal{H}\}$$

3. the spherical continuous spectrum of T :

$$\sigma_{cS}(T) := \{s \in \mathbb{H} : \ker(Q_s(T)) = \{0\}, \overline{\operatorname{ran}(Q_s(T))} = \mathcal{H}, Q_s(T)^{-1} \notin \mathcal{B}(\mathcal{H})\}.$$

Theorem 2.23. Let $T \in \mathcal{B}(\mathcal{H})$. Then $s \in \sigma_{pS}(T)$ if and only if s is a right eigenvalue of T , i.e. there exists $x \in \mathcal{H}$ such that $Tx = xs$.

The adjoint of an operator and related properties are defined analogue to the complex case.

Definition 2.24. Let $T \in \mathcal{B}(\mathcal{H})$. The adjoint T^* of T is the unique operator that satisfies

$$\langle T^*x, y \rangle = \langle x, Ty \rangle \quad \forall x, y \in \mathcal{H}.$$

Definition 2.25. An operator $T \in \mathcal{B}(\mathcal{H})$ is called

- (i) selfadjoint, if $T^* = T$
- (ii) anti-selfadjoint, if $T^* = -T$
- (iii) positive if $\langle x, Tx \rangle \geq 0$ for all $x \in \mathcal{H}$
- (iv) normal if $T^*T = TT^*$
- (v) unitary if $T^* = T^{-1}$.

If $T \in \mathcal{B}(\mathcal{H})$ is positive, then there exists a unique positive operator S such that $S^2 = T$. We denote $\sqrt{T} := S$. Moreover, for any operator $T \in \mathcal{B}(\mathcal{H})$, the operator T^*T is positive and we can define

$$|T| := \sqrt{T^*T},$$

which allows to prove the polar decomposition theorem.

Theorem 2.26. Let \mathcal{H} be a quaternionic Hilbert space and let $T \in \mathcal{B}(\mathcal{H})$. Then there exist two unique operators W and P in $\mathcal{B}(\mathcal{H})$ such that

- (i) $T = WP$
- (ii) $P \geq 0$
- (iii) $\ker(P) \subset \ker(W)$
- (iv) $\forall u \in \ker(P)^\perp : \|Wu\| = \|u\|$.

The operators W and P have the following properties

- (a) $P = |T|$
- (b) If T is normal then W defines a unitary operator in $\mathcal{B}(\overline{\text{ran}(T)})$
- (c) W is (anti) self-adjoint if T is.

The above theorem is mentioned several times in the literature, for a proof see [28]. In order to investigate linear operators on a quaternionic Hilbert space, it is often useful to define a complex structure on the space that allows to write every vector in terms of two components that belong to a certain complex Hilbert space. If this complex structure is chosen appropriately, then the considered operator is the natural extension of a complex linear operator on the component space.

Definition 2.27 (See G. Emch [23]). Let $J \in \mathcal{B}(\mathcal{H})$ be anti-selfadjoint and unitary and let $i \in \mathbb{S}$. We define the complex subspaces

$$\mathcal{H}_+^{Ji} := \{x \in \mathcal{H} : Jx = xi\} \quad \text{and} \quad \mathcal{H}_-^{Ji} := \{x \in \mathcal{H} : Jx = -xi\}.$$

Theorem 2.28. \mathcal{H}_+^{Ji} is a nontrivial complex Hilbert space over \mathbb{C}_i with respect to the structure induced by \mathcal{H} : its sum is the sum of \mathcal{H} , its complex scalar multiplication is the right scalar multiplication of \mathcal{H} restricted to \mathbb{C}_i and its complex scalar product is the restriction of the scalar product of \mathcal{H} to $\mathcal{H}_+^{Ji} \times \mathcal{H}_+^{Ji}$. An analogous statement holds true for \mathcal{H}_-^{Ji} .

Theorem 2.29. Every orthonormal basis $(e_n)_{n \in \mathbb{N}}$ of the complex Hilbert space \mathcal{H}_+^{Ji} is also an orthonormal basis of \mathcal{H} . Moreover, it is

$$Jx = \sum_{n \in \mathbb{N}} e_n i \langle e_n, x \rangle, \quad \forall x \in \mathcal{H}.$$

For a proof of the above two results see [28]. Observe that Theorem 2.29 implies that we can write $x \in \mathcal{H}$ as $x = x_1 + x_2 j$ with $x_1, x_2 \in \mathcal{H}_+^{Ji}$. Moreover, it justifies considering J as a left multiplication with the imaginary unit $i \in \mathbb{S}$. We may set $ix := Jx$ for any $x \in \mathcal{H}$ and extend this left scalar multiplication to the entire complex field \mathbb{C}_i by setting $(a_0 + ia_1)x = xa_0 + Jxa_1$. Since

$$ix = Jx = Jx_0 + Jx_1 j = x_0 i + x_1 i j,$$

we then obtain $ax = x_0 a + x_1 a j$ for $a \in \mathbb{C}_i$.

Note that the choice of the imaginary unit i is arbitrary, but that different imaginary units will lead to different left scalar multiplications, which are of course only defined for scalars in the respective complex plane!

We discuss now the relation between quaternionic linear operators on \mathcal{H} and complex linear operators on \mathcal{H}_+^{Ji} . In order to avoid confusion, we distinguish complex linear operators by a \mathbb{C} -subscript.

Theorem 2.30. Let J be an anti-selfadjoint, unitary operator on \mathcal{H} and take $i \in \mathbb{S}$. For every bounded \mathbb{C}_i -linear operator $T_{\mathbb{C}} \in \mathcal{B}(\mathcal{H}_+^{Ji})$, there exists a unique right \mathbb{H} -linear operator $\widetilde{T}_{\mathbb{C}} \in \mathcal{B}(\mathcal{H})$ that satisfies

$$\widetilde{T}_{\mathbb{C}} x = T_{\mathbb{C}} x, \quad \forall x \in \mathcal{H}_+^{Ji}.$$

The following facts also hold:

- (a) We have $\|\widetilde{T}_{\mathbb{C}}\| = \|T_{\mathbb{C}}\|$.
 - (b) It is $[\widetilde{T}_{\mathbb{C}}, J] = 0$, where $[\widetilde{T}_{\mathbb{C}}, J] := \widetilde{T}_{\mathbb{C}} J - J \widetilde{T}_{\mathbb{C}}$ denotes the commutator of $\widetilde{T}_{\mathbb{C}}$ and J .
 - (c) An operator $V \in \mathcal{B}(\mathcal{H})$ is equal to $\widetilde{T}_{\mathbb{C}}$ for some \mathbb{C}_i -linear operator $T_{\mathbb{C}} \in \mathcal{B}(\mathcal{H}_+^{Ji})$ if and only if $[J, V] = 0$.
 - (d) We have $(\widetilde{T}_{\mathbb{C}})^* = \widetilde{T}_{\mathbb{C}}^*$.
- Furthermore, given another \mathbb{C}_i -linear operator $S_{\mathbb{C}} \in \mathcal{B}(\mathcal{H}_+^{Ji})$, we have
- (e) $\widetilde{S_{\mathbb{C}} T_{\mathbb{C}}} = \widetilde{S_{\mathbb{C}}} \widetilde{T_{\mathbb{C}}}$.
 - (f) If $S_{\mathbb{C}}$ is a right (resp. left) inverse of $T_{\mathbb{C}}$, then $\widetilde{S_{\mathbb{C}}}$ is a right (resp. left) inverse of $\widetilde{T_{\mathbb{C}}}$.

The above theorem generalizes some results of G. Emch in Section 3 of [23] and formulates them in a modern language.

Definition 2.31. For an anti-selfadjoint unitary operator J , we define the set

$$\mathcal{B}_J(\mathcal{H}) := \{T \in \mathcal{B}(\mathcal{H}) : [T, J] = 0\}.$$

Remark 2.32. For $T \in \mathcal{B}_J(\mathcal{H})$ and $a = a_0 + ia_1 \in \mathbb{C}_i$ set $aT = a_0 T + a_1 J T$. If we consider J as a left-multiplication with some imaginary unit $i \in \mathbb{S}$, then $\mathcal{B}_J(\mathcal{H})$ turns into a Banach algebra over \mathbb{C}_i that is isometrically isomorphic to $\mathcal{B}(\mathcal{H}_+^{Ji})$. Isometric isomorphisms between these two spaces are given by

$$\text{Res}_{Ji} : \begin{cases} \mathcal{B}_J(\mathcal{H}) & \rightarrow \mathcal{B}(\mathcal{H}_+^{Ji}) \\ T & \mapsto T|_{\mathcal{H}_+^{Ji}} \end{cases} \quad \text{Lift}_{Ji} : \begin{cases} \mathcal{B}(\mathcal{H}_+^{Ji}) & \rightarrow \mathcal{B}_J(\mathcal{H}) \\ T_{\mathbb{C}} & \mapsto \widetilde{T_{\mathbb{C}}} \end{cases}, \quad (6)$$

where $\widetilde{T_{\mathbb{C}}}$ is the unique operator obtained from Theorem 2.30.

Corollary 2.33. *An operator $T \in \mathcal{B}_J(\mathcal{H})$ is selfadjoint, anti-selfadjoint, positive, normal or unitary if and only if $\text{Res}_{Ji}(T)$ is.*

3 Schatten classes of quaternionic linear operators

In order to introduce Schatten classes of quaternionic linear operators, we need to define the singular values of compact quaternionic linear operators. For analogue results in the complex case see [38].

3.1 Singular values of compact operators

Definition 3.1. A right linear operator is called compact if it maps bounded sequences to sequences that admit convergent subsequences. We denote the set of all compact quaternionic right linear operators on \mathcal{H} by $\mathcal{B}_0(\mathcal{H})$.

We recall the spectral theorem for compact quaternionic operators (see [29]); note that this is a special case of the spectral theorem for arbitrary normal operators, which was recently established in [3].

Theorem 3.2. *Given a normal operator $T \in \mathcal{B}_0(\mathcal{H})$ with spherical point spectrum $\sigma_{pS}(T)$ there exists a Hilbert basis $\mathcal{N} \subset \mathcal{H}$ made of eigenvectors of T such that*

$$Tx = \sum_{z \in \mathcal{N}} z \lambda_z \langle z, x \rangle \quad \forall x \in \mathcal{H}, \quad (7)$$

where $\lambda_z \in \mathbb{H}$ is an eigenvalue relative to the eigenvector z and if $\lambda_z \neq 0$ then there are only a finite number of distinct $z' \in \mathcal{N}$ such that $\lambda_z = \lambda_{z'}$. Moreover the values λ_z are at most countably many and 0 is their only possible accumulation point.

Recall that we are considering only separable Hilbert spaces. Hence, \mathcal{N} is always countable and we can write $\mathcal{N} = \{e_n : n \in \mathbb{N}\}$.

Remark 3.3. Observe that the eigenvalues $(\lambda_n)_{n \in \mathbb{N}}$ in the spectral decomposition of a compact quaternionic linear operator are in general not unique. Indeed, if $Te_n = e_n \lambda_n$ and $q \in \mathbb{H}$ with $|q| = 1$, then

$$T(e_n q) = (Te_n)q = e_n \lambda_n q = (e_n q) q^{-1} \lambda_n q$$

and hence the eigenpair (e_n, λ_n) can be replaced by $(e_n q, q^{-1} \lambda_n q)$. However, $q^{-1} \lambda_n q \in [\lambda_n]$ and consequently at least the eigenspheres $[\lambda_n]$ are uniquely determined. In particular, the eigenvalues can be chosen such that $\lambda_n \in \mathbb{C}_i^+ := \{s = s_0 + i s_1 \in \mathbb{C}_i : s_1 \geq 0\}$ for a given imaginary unit $i \in \mathbb{S}$, cf. Lemma 2.1.

Remark 3.4. Using the spectral theorem, one can define a functional calculus for so-called slice functions. We just need the special case of fractional powers of a positive operator compact operator T : consider its spectral decomposition $T = \sum_{n \in \mathbb{N}} e_n \lambda_n \langle e_n, \cdot \rangle$. For $p > 0$, the operator T^p is defined as

$$T^p = \sum_{n \in \mathbb{N}} e_n \lambda_n \langle e_n, \cdot \rangle.$$

Consider now an arbitrary compact operator T . Then the operator $|T|$ is normal and, combining Theorem 3.2 applied to $|T|$ with the polar decomposition, Theorem 2.26, we are capable of finding a Hilbert-basis $(e_n)_{n \in \mathbb{N}}$ and an orthonormal set $(\sigma_n)_{n \in \mathbb{N}}$ in \mathcal{H} such that

$$Tx = \sum_{n \in \mathbb{N}} \sigma_n \lambda_n \langle e_n, x \rangle \quad \forall x \in \mathcal{H}, \quad (8)$$

where the $\lambda_n \in \mathbb{R}^+$ are the eigenvalues of the operator $|T|$ in decreasing order, the vectors $(e_n)_{n \in \mathbb{N}}$ form an eigenbasis of $|T|$ and $\sigma_n = W e_n$ with W unitary such that $T = W|T|$.

Definition 3.5. We call the set $\{\lambda_n\}_{n \in \mathbb{N}}$ the set of singular values of T and the representation (8) the singular value decomposition of T .

The following Rayleigh's equation gives a characterization of the singular values of T . Since the proof follows the lines of the complex case, we omit it and refer to [22].

Lemma 3.6. *Let T be a positive, compact operator on \mathcal{H} and let*

$$Tx = \sum_{n \in \mathbb{N}} e_n \lambda_n \langle e_n, x \rangle \quad (9)$$

be the spectral decomposition of T , where the eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}}$ are given in descending order. Then

$$\lambda_{n+1} = \min_{y_1, \dots, y_n} \max_{\substack{\langle x, y_i \rangle = 0 \\ i=1, \dots, n}} \frac{\langle x, Tx \rangle}{\|x\|^2}.$$

Also in the quaternionic setting we have:

Lemma 3.7. *Let T be a positive, compact operator on \mathcal{H} . Then*

$$\lambda_{n+1} = \min_{y_1, \dots, y_n} \max_{\substack{\langle x, y_i \rangle = 0 \\ i=1, \dots, n \\ \|x\|=1}} \|Tx\|. \quad (10)$$

Proof. In fact

$$\lambda_{n+1} = \min_{y_1, \dots, y_n} \max_{\substack{\langle x, y_i \rangle = 0 \\ i=1, \dots, n \\ \|x\|=1}} \langle x, Tx \rangle.$$

using the Cauchy-Schwarz inequality we see that

$$\langle x, Tx \rangle \leq \|x\| \|Tx\|$$

and since we assumed in the above lemma that T is positive, we can consider its spectral decomposition (9) in order to see that

$$\begin{aligned} \lambda_{n+1} &= \min_{y_1, \dots, y_n} \max_{\substack{\langle x, y_i \rangle = 0 \\ i=1, \dots, n \\ \|x\|=1}} \langle x, Tx \rangle \leq \min_{y_1, \dots, y_n} \max_{\substack{\langle x, y_i \rangle = 0 \\ i=1, \dots, n \\ \|x\|=1}} \|Tx\| \leq \max_{\substack{\langle x, e_i \rangle = 0 \\ i=1, \dots, n \\ \|x\|=1}} \|Tx\| \\ &= \max_{\substack{\langle x, e_i \rangle = 0 \\ i=1, \dots, n \\ \|x\|=1}} \sqrt{\sum_{m \geq n+1} \lambda_m^2 |\langle x, e_m \rangle|^2} \leq \lambda_{n+1} \max_{\substack{\langle x, e_i \rangle = 0 \\ i=1, \dots, n \\ \|x\|=1}} \sqrt{\sum_{m \geq n+1} |\langle x, e_m \rangle|^2} \\ &\leq \lambda_{n+1}. \end{aligned}$$

□

The singular values of an arbitrary compact operator T are the eigenvalues of the positive operator $|T|$. Since the eigenvalues of $|T|$ can be obtained via the formula (10) and since $\|Tx\| = \|W|T|x\| = \||T|x\|$ by Theorem 2.26, the singular values of T also satisfy (10).

The formula (10) also allows us to deduce the following corollary, just as in the complex case, cf. [38, Theorem 1.34].

Corollary 3.8. *Let T be a compact operator on \mathcal{H} . Its singular values satisfy*

$$\lambda_{n+1} = \inf_{F \in \mathcal{F}_n} \|T - F\|, \quad (11)$$

where \mathcal{F}_n is the set of all linear operators on \mathcal{H} with rank less than or equal to n .

The following proposition is an immediate consequence of (10), cf. [22, Corollary XI.9.3] for the complex case.

Corollary 3.9. *Let T_1 and T_2 be compact linear operators on \mathcal{H} . Then, for every nonnegative $n, m \in \mathbb{N}_0$, we have that*

$$\lambda_{n+m+1}(T_1 + T_2) \leq \lambda_{n+1}(T_1) + \lambda_{m+1}(T_2)$$

and

$$\lambda_{n+m+1}(T_1 T_2) \leq \lambda_{n+1}(T_1) \lambda_{m+1}(T_2).$$

3.2 Definition of the Schatten class

For $p \in (0, +\infty)$ the Schatten p -class $S_p(\mathcal{H}_{\mathbb{C}})$ of operators on a complex Hilbert space $\mathcal{H}_{\mathbb{C}}$ is defined as the set of compact operators whose singular value sequences are p -summable. For $p \geq 1$, the Schatten p -norm, which assigns to each operator the ℓ^p -norm of its singular value sequence, turns $S_p(\mathcal{H}_{\mathbb{C}})$ into a Banach space. However, the analogue approach does not make sense in the quaternionic setting. Major problems appear in particular when trying to define the trace of an operator, which plays a crucial role in the classical theory. The trace of a compact complex linear operator is defined as

$$\mathrm{tr} A = \sum_{n \in \mathbb{N}} \langle e_n, A e_n \rangle, \quad (12)$$

where $(e_n)_{n \in \mathbb{N}}$ is an orthonormal basis of the Hilbert space $\mathcal{H}_{\mathbb{C}}$. It can be shown that the trace is independent of the choice of the orthonormal basis and therefore well-defined.

In general, this is not true in the quaternionic setting. Consider for example a compact normal operator T and its spectral decompositions

$$T = \sum_{n \in \mathbb{N}} e_n \lambda_n \langle e_n, \cdot \rangle = \sum_{n \in \mathbb{N}} \tilde{e}_n \tilde{\lambda}_n \langle \tilde{e}_n, \cdot \rangle,$$

where the orthonormal system $(e_n)_{n \in \mathbb{N}}$ is such that the corresponding eigenvalues $(\lambda_n)_{n \in \mathbb{N}}$ belong to the complex halfplane \mathbb{C}_i^+ and the orthonormal system $(\tilde{e}_n)_{n \in \mathbb{N}}$ is such that the corresponding eigenvalues $(\tilde{\lambda}_n)_{n \in \mathbb{N}}$ belong to the complex halfplane \mathbb{C}_j^+ with $i, j \in \mathbb{S}$ and $i \neq j$, cf. Remark 3.3. Moreover, assume that at least one eigenvalue has nonzero imaginary part. Then

$$\sum_{n \in \mathbb{N}} \langle T e_n, e_n \rangle = \sum_{n \in \mathbb{N}} \lambda_n \langle e_n, e_n \rangle = \sum_{n \in \mathbb{N}} \lambda_n \in \mathbb{C}_i^+ \setminus \mathbb{R},$$

but

$$\sum_{n \in \mathbb{N}} \langle T \tilde{e}_n, \tilde{e}_n \rangle = \sum_{n \in \mathbb{N}} \tilde{\lambda}_n \langle \tilde{e}_n, \tilde{e}_n \rangle = \sum_{n \in \mathbb{N}} \tilde{\lambda}_n \in \mathbb{C}_j^+ \setminus \mathbb{R}.$$

Obviously, it is

$$\sum_{n \in \mathbb{N}} \langle T e_n, e_n \rangle \neq \sum_{n \in \mathbb{N}} \langle T \tilde{e}_n, \tilde{e}_n \rangle.$$

In contrast to the classical theory, we have to restrict ourselves to operators that satisfy an additional restriction: compatibility with a chosen complex left-multiplication.

Definition 3.10. Let $J \in \mathcal{B}(\mathcal{H})$ be an anti-selfadjoint and unitary operator. For $p \in (0, +\infty]$, we define the (J, p) -Schatten class of operators $S_p(J)$ as

$$S_p(J) := \{T \in \mathcal{B}_0(\mathcal{H}) : [T, J] = 0 \text{ and } (\lambda_n(T))_{n \in \mathbb{N}} \in \ell^p\},$$

where $(\lambda_n(T))_{n \in \mathbb{N}}$ denotes the sequence of singular values of T and ℓ^p denotes the space of p -summable resp. bounded sequences. For $T \in S_p(J)$, we define

$$\|T\|_p = \left(\sum_{n \in \mathbb{N}} |\lambda_n(T)|^p \right)^{\frac{1}{p}} \quad \text{if } p \in (0, +\infty) \quad (13)$$

and

$$\|T\|_p = \sup_{n \in \mathbb{N}} \lambda_n(T) = \|T\| \quad \text{if } p = +\infty.$$

Obviously, $S_p(J)$ is a subset of $\mathcal{B}_J(\mathcal{H})$, see Definition 2.31. The following lemma shows that it is isometrically isomorphic to the Schatten p -class on \mathcal{H}_+^{Ji} if J is considered as a left-multiplication with $i \in \mathbb{S}$.

Lemma 3.11. *Let J be a unitary and anti-selfadjoint operator and let $i \in \mathbb{S}$. For all $T \in \mathcal{B}_0(\mathcal{H})$ such that $[T, J] = 0$ and for each $n \in \mathbb{N}$, we have that*

$$\lambda_n(T) = \lambda'_n(\text{Res}_{Ji}(T)),$$

where $\lambda'_n(A_{\mathbb{C}})$ is the n -th singular value of a \mathbb{C}_i -linear operator $A_{\mathbb{C}} \in \mathcal{B}(\mathcal{H}_+^{Ji})$.

Proof. Since T is compact, its restriction $\text{Res}_{Ji}(T) : \mathcal{H}_+^{Ji} \rightarrow \mathcal{H}_+^{Ji}$ is a compact operator on \mathcal{H}_+^{Ji} . Hence, we can find orthonormal sets $(e_n)_{n \in \mathbb{N}}$ and $(\sigma_n)_{n \in \mathbb{N}}$ in \mathcal{H}_+^{Ji} such that

$$Tx = \sum_{n \in \mathbb{N}} \sigma_n \lambda'_n \left(T|_{\mathcal{H}_+^{Ji}} \right) \langle e_n, x \rangle \quad \forall x \in \mathcal{H}_+^{Ji}.$$

By the uniqueness of the extension, it follows that this expression holds for all $x \in \mathcal{H}$ and we deduce $\lambda_n(T) = \lambda'_n(\text{Res}_{Ji}(T))$. □

Corollary 3.12. *Let J be an anti-selfadjoint unitary operator on \mathcal{H} and consider it as a left-multiplication with $i \in \mathbb{S}$. For any $p \in (0, +\infty]$, the space $S_p(J)$ is isomorphic to $S_p(\mathcal{H}_+^{Ji})$, the Schatten p -class on \mathcal{H}_+^{Ji} . An isomorphism between these spaces is given by $T \mapsto \text{Res}_{Ji}(T)$. Moreover, $\|T\|_p = \|\text{Res}_{Ji}(T)\|_p$. In particular, if $p \in [1, +\infty]$, then $\|\cdot\|_p$ is actually a norm and $S_p(J)$ is a Banach space over \mathbb{C}_i that is even isometrically isomorphic to $S_p(\mathcal{H}_+^{Ji})$.*

Finally, as an immediate consequence of Corollary 3.9, we obtain as in the complex case that any $S_p(J)$ is an ideal of $\mathcal{B}_J(\mathcal{H})$.

Corollary 3.13. *Let $p \in (0, +\infty]$. Then $S_p(J)$ is a two-sided ideal of $\mathcal{B}_J(\mathcal{H})$, i.e. ST and TS belong to $S_p(J)$ whenever $T \in S_p(J)$ and $S \in \mathcal{B}_J(\mathcal{H})$.*

3.3 The dual of the Schatten class

In the following, we fix a unitary, anti-selfadjoint operator J and consider it as a left-multiplication with $i \in \mathbb{S}$. We define the trace of an operator $T \in S_1(J)$ and establish some elementary results in order to determine the dual space of $S_p(J)$.

Definition 3.14. We define the Ji -trace of an operator $T \in S_1(J)$ as

$$\text{Tr}_{Ji}(T) := \text{tr}(\text{Res}_{Ji}(T))$$

where $\text{tr}(\text{Res}_{Ji}(T))$ denotes the classical trace of a complex linear operator $\text{Res}_{Ji}(T)$ as defined in (12).

Corollary 3.15. *If $T \in S_1(J)$, then*

$$\mathrm{Tr}_{Ji}(T) = \sum_{n \in \mathbb{N}} \langle e_n, T e_n \rangle \quad (14)$$

for any orthonormal basis $(e_n)_{n \in \mathbb{N}}$ of \mathcal{H}_+^{Ji} .

If T is positive or selfadjoint, then also in the quaternionic setting there are no restrictions on the choice of the orthonormal basis in (14).

Lemma 3.16. *If T is a positive and compact operator on \mathcal{H} with singular values $(\lambda_n)_{n \in \mathbb{N}}$ then*

$$\sum_{n \in \mathbb{N}} \lambda_n = \sum_{n \in \mathbb{N}} \langle e_n, T e_n \rangle$$

for each orthonormal basis $(e_n)_{n \in \mathbb{N}}$ of \mathcal{H} . If in particular $T \in S_1(J)$, then (14) holds true for any orthonormal basis of \mathcal{H} .

Proof. Since T is compact and positive, we have $T = \sum_{n \in \mathbb{N}} \eta_n \lambda_n \langle \eta_n, \cdot \rangle$ for some orthonormal basis $(\eta_n)_{n \in \mathbb{N}}$ of \mathcal{H} by Theorem 3.2. For any orthonormal basis $(e_n)_{n \in \mathbb{N}}$ of \mathcal{H} , we therefore have

$$\langle e_n, T e_n \rangle = \sum_{m \in \mathbb{N}} \lambda_m |\langle \eta_m, e_n \rangle|^2.$$

Using Fubini's theorem and Parseval's identity $\|x\|^2 = \sum_{n \in \mathbb{N}} |\langle e_n, x \rangle|^2$, we are left with

$$\begin{aligned} \sum_{n \in \mathbb{N}} \langle e_n, T e_n \rangle &= \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} \lambda_m |\langle \eta_m, e_n \rangle|^2 \\ &= \sum_{m \in \mathbb{N}} \lambda_m \sum_{n \in \mathbb{N}} |\langle \eta_m, e_n \rangle|^2 = \sum_{m \in \mathbb{N}} \lambda_m. \end{aligned}$$

□

Corollary 3.17. *Let $T \in S_1(J)$ be selfadjoint. Then (14) holds true for any orthonormal basis of \mathcal{H} .*

Proof. By Corollary 2.33, the operator $\mathrm{Res}_{Ji}(T)$ is a bounded selfadjoint operator on \mathcal{H}_+^{Ji} and can therefore be decomposed into $\mathrm{Res}_{Ji}(T) = T_{+, \mathbb{C}} - T_{-, \mathbb{C}}$ with positive operators $T_{+, \mathbb{C}}, T_{-, \mathbb{C}} \in \mathcal{B}(\mathcal{H}_+^{Ji})$. If we set $T_{\pm} = \mathrm{Lift}_{Ji}(T_{\pm, \mathbb{C}})$, then $T = T_+ - T_-$ with positive operators $T_+, T_- \in \mathcal{B}_J(\mathcal{H})$. The additivity of the Ji -trace and Lemma 3.16 imply

$$\begin{aligned} \mathrm{Tr}_{Ji}(T) &= \mathrm{Tr}_{Ji}(T_+) - \mathrm{Tr}_{Ji}(T_-) \\ &= \sum_{n \in \mathbb{N}} \langle e_n, T_+ e_n \rangle - \sum_{n \in \mathbb{N}} \langle e_n, T_- e_n \rangle = \sum_{n \in \mathbb{N}} \langle e_n, T e_n \rangle \end{aligned}$$

for any orthonormal basis $(e_n)_{n \in \mathbb{N}}$ of \mathcal{H} .

□

Observe that $\mathrm{Tr}_{Ji}(T)$ of course depends on the imaginary unit $i \in \mathbb{S}$. However, as the next result shows, the choice of the imaginary unit i has no essential impact.

Lemma 3.18. *Let $T \in S_1(J)$ and let i, j in \mathbb{S} . Then*

$$\mathrm{Tr}_{Jj}(T) = \phi(\mathrm{Tr}_{Ji}(T)),$$

where ϕ is the isomorphism $\phi(z_0 + iz_1) = z_0 + jz_1$ between the complex fields \mathbb{C}_i and \mathbb{C}_j .

Proof. Let $(e_n)_{n \in \mathbb{N}}$ be an orthonormal basis of $\mathcal{H}_+^{J_i}$ and let $q \in \mathbb{H}$ with $|q| = 1$ such that $j = q^{-1}iq$, cf. Lemma 2.1. Then $\phi(z) = q^{-1}zq$ for any $z \in \mathbb{C}_i$. Moreover, $(e_nq)_{n \in \mathbb{N}}$ is an orthonormal basis of $\mathcal{H}_+^{J_j}$ as

$$J(e_nq) = (Je_n)q = e_niq = (e_nq)q^{-1}iq = (e_nq)j.$$

Since $|q| = 1$, we have $q^{-1} = \bar{q}$ and thus, by Corollary 3.15, we obtain

$$\phi(\text{Tr}_{J_i}(T)) = q^{-1} \text{Tr}_{J_i}(T)q = \sum_{n=0}^{+\infty} \langle e_nq, Te_nq \rangle = \text{Tr}_{J_j}(T).$$

□

Lemma 3.19. *Suppose $1 \leq p < +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $T \in S_p(J)$ and $S \in S_q(J)$, then*

(i) *TS and ST belong to the trace class $S_1(J)$*

(ii) *$\text{Tr}_{J_i}(TS) = \text{Tr}_{J_i}(ST)$*

(iii) *$|\text{Tr}_{J_i}(TS)| \leq \|T\|_p \|S\|_q$.*

Proof. Applying Corollary 3.12, we can reduce the statement to the case of operators on a complex Hilbert space, where we know that these results are true. Hence,

$$\begin{aligned} & T \in S_p(J) \text{ and } S \in S_q(J) \\ \iff & \text{Res}_{J_i}(T) \in S_p(\mathcal{H}_+^{J_i}) \text{ and } \text{Res}_{J_i}(S) \in S_q(\mathcal{H}_+^{J_i}) \\ \implies & \text{Res}_{J_i}(TS) \in S_1(\mathcal{H}_+^{J_i}) \\ \iff & TS \in S_1(J). \end{aligned}$$

The second statement follows from the definition:

$$\begin{aligned} \text{Tr}_{J_i}(TS) &= \text{tr}(\text{Res}_{J_i}(TS)) \\ &= \text{tr}(\text{Res}_{J_i}(T) \text{Res}_{J_i}(S)) \\ &= \text{tr}(\text{Res}_{J_i}(S) \text{Res}_{J_i}(T)) \\ &= \text{tr}(\text{Res}_{J_i}(ST)) \\ &= \text{Tr}_{J_i}(ST). \end{aligned}$$

The final statement relies on the fact that Res_{J_i} and Lift_{J_i} are p -norm preserving:

$$\begin{aligned} |\text{Tr}_{J_i}(TS)| &= |\text{tr}(\text{Res}_{J_i}(TS))| \\ &= |\text{tr}(\text{Res}_{J_i}(T) \text{Res}_{J_i}(S))| \\ &\leq \|\text{Res}_{J_i}(T)\|_p \|\text{Res}_{J_i}(S)\|_q \\ &= \|T\|_p \|S\|_q, \end{aligned}$$

which finishes the proof of the lemma.

□

Lemma 3.20. *Suppose $1 \leq p < +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $T \in S_p(J)$, then*

$$\|T\|_p = \sup \{ |\text{tr}_{J_i}(ST)| : \|S\|_q = 1, S \in S_q(J) \}.$$

Proof. Using the fact that this result is true for operators on a complex Hilbert space and that Res_{J_i} is a p -norm preserving isomorphism, we find that

$$\begin{aligned}\|T\|_p &= \|\text{Res}_{J_i}(T)\|_p \\ &= \sup \{ |\text{tr}((S_{\mathbb{C}} \text{Res}_{J_i}(T)))| : \|S_{\mathbb{C}}\|_q = 1, S_{\mathbb{C}} \in S_q(\mathcal{H}_+^{J_i}) \} \\ &= \sup \{ |\text{tr}((\text{Res}_{J_i}(ST)))| : \|\text{Res}_{J_i}(S)\|_q = 1, S \in S_q(J) \} \\ &= \sup \{ |\text{Tr}_{J_i}(ST)| : \|S\|_q = 1, S \in S_q(J) \}.\end{aligned}$$

□

We can also use the fact that $S_p(J)$ is isometrically isomorphic to the Schatten p -class of operators on the complex Hilbert space $\mathcal{H}_+^{J_i}$ to determine its dual space.

Theorem 3.21. *If $1 \leq p < +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$S_p(J)^* = S_q(J)$$

with equal norms and under the pairing $\langle T, S \rangle = \text{Tr}_{J_i}(TS)$.

Proof. First, assume that ξ is a continuous linear functional on $S_p(J)$. By Corollary 3.12, the mapping $T_{\mathbb{C}} \mapsto \xi(\text{Lift}_{J_i}(T_{\mathbb{C}}))$ is a linear functional on $S_p(\mathcal{H}_+^{J_i})$ with $\|\xi \circ \text{Lift}_{J_i}\| = \|\xi\|$.

Since $S_p(\mathcal{H}_+^{J_i})^* \cong S_q(\mathcal{H}_+^{J_i})$ under the pairing $\langle T_{\mathbb{C}}, S_{\mathbb{C}} \rangle = \text{tr}(T_{\mathbb{C}} S_{\mathbb{C}})$, there exists an operator $S_{\mathbb{C}} \in S_q(\mathcal{H}_+^{J_i})$ with $\|S_{\mathbb{C}}\|_q = \|\xi \circ \text{Lift}_{J_i}\|$ such that

$$\xi \circ \text{Lift}_{J_i}(T_{\mathbb{C}}) = \text{tr}(T_{\mathbb{C}} S_{\mathbb{C}}), \quad \forall T_{\mathbb{C}} \in \mathcal{H}_+^{J_i}.$$

We define now $S_{\xi} := \text{Lift}_{J_i}(S_{\mathbb{C}})$. Then

$$\|S_{\xi}\|_q = \|S_{\mathbb{C}}\|_q = \|\xi \circ \text{Lift}_{J_i}\| = \|\xi\|$$

and

$$\begin{aligned}\xi(T) &= \xi \circ \text{Lift}_{J_i} \circ \text{Res}_{J_i}(T) \\ &= \text{tr}(\text{Res}_{J_i}(T) S_{\mathbb{C}}) \\ &= \text{tr}(\text{Res}_{J_i}(TS_{\xi})) \\ &= \text{Tr}_{J_i}(TS_{\xi}).\end{aligned}$$

Hence, the mapping $\Phi : \xi \mapsto S_{\xi}$ is an isometric \mathbb{C}_i -linear mapping of $S_p(J)^*$ into $S_q(J)$.

Conversely, it follows from the \mathbb{C}_i -linearity of the J_i -trace and (iii) of Lemma 3.19 that, for $S \in S_q(J)$, the mapping $T \mapsto \xi_S(T) := \text{Tr}_{J_i}(TS)$ is a bounded \mathbb{C}_i -linear functional on $S_p(J)$ with $\|\xi_S\| \leq \|S\|_q$. Consequently, Φ is even invertible and in turn $S_p(J)^*$ is isometrically isomorphic to $S_q(J)$.

□

3.4 Characterizations of Schatten class operators

In this section we take a closer look at the singular values of Schatten class operators in order to arrive at necessary and sufficient conditions for an operator to belong to the Schatten class. A crucial observation was made in Corollary 3.12:

$$T \in S_p(J) \iff \text{Res}_{J_i}(T) \in S_p(\mathcal{H}_+^{J_i}).$$

Lemma 3.22. *Let T be a positive and compact operator on \mathcal{H} and $p \in (0, +\infty)$. Then*

$$T \in S_p \iff T^p \in S_1.$$

Moreover, $\|T\|_p^p = \|T^p\|_1$.

Proof. Let $T = \sum_{n \in \mathbb{N}} e_n \langle e_n, \cdot \rangle \lambda_n$ be a spectral decomposition of T . Since the eigenvalues λ_n are positive, they coincide with the singular values of T . Moreover, as in the case of complex operators, we have $\sigma_S(T) = \{\lambda_n : n \in \mathbb{N}\} \cup \{0\}$, cf. [25]. The S -spectral theorem implies that $\sigma_S(T^p) = \{\lambda_n^p : n \in \mathbb{N}\} \cup \{0\}$. Consequently, the singular value sequence of T^p is $(\lambda_n^p)_{n \in \mathbb{N}}$, and hence

$$\|T^p\|_1 = \sum_{n \in \mathbb{N}} \lambda_n^p = \|T\|_p^p.$$

□

Theorem 3.23. *If T is a compact operator on \mathcal{H} such that $[T, J] = 0$ and $p \in (0, +\infty)$ then*

$$T \in S_p(J) \iff |T|^p = (T^*T)^{\frac{p}{2}} \in S_1(J) \iff T^*T \in S_{\frac{p}{2}}(J).$$

Moreover

$$\|T\|_p^p = \| |T| \|_p^p = \| |T|^p \|_1 = \|T^*T\|_{\frac{p}{2}}^{\frac{p}{2}}.$$

As a consequence, we have that

$$T \in S_p(J) \iff |T| \in S_p(J).$$

Proof. We start by proving the first equivalence (the other ones follow analogously from the corresponding results for complex linear operators, cf. [38, Theorem 1.26]):

$$\begin{aligned} T \in S_p(J) &\iff \text{Res}_{J_i}(T) \in S_p(\mathcal{H}_+^{J_i}) \\ &\iff |\text{Res}_{J_i}(T)|^p = \text{Res}_{J_i}(|T|^p) \in S_1(\mathcal{H}_+^{J_i}) \\ &\iff |T|^p \in S_p(J). \end{aligned}$$

For the equality of the norms we prove the first one (the rest is proven in a similar way):

$$\|T\|_p^p = \|\text{Res}_{J_i}(T)\|_p^p = \| |\text{Res}_{J_i}(T)| \|_p^p = \|\text{Res}_{J_i}(|T|)\|_p^p = \| |T| \|_p^p.$$

□

Since Res_{J_i} and Lift_{J_i} are p -norm preserving, we easily obtain further characterizations of (J, p) -Schatten class operators from the respective results in the complex case, cf. [38, Theorems 1.27–1.29].

Theorem 3.24. *Suppose that T is a compact operator on a quaternionic Hilbert-space \mathcal{H} with $[T, J] = 0$ and that $p \geq 1$. Then T is in $S_p(J)$ if and only if*

$$\sum_{n \in \mathbb{N}} |\langle e_n, T e_n \rangle|^p < +\infty$$

for all orthonormal sets $(e_n)_{n \in \mathbb{N}}$ in $\mathcal{H}_+^{J_i}$. If T is also selfadjoint then

$$\|T\|_p = \sup \left\{ \left[\sum_{n \in \mathbb{N}} |\langle e_n, T e_n \rangle|^p \right]^{\frac{1}{p}} : \{e_n\} \text{ orthonormal set in } \mathcal{H}_+^{J_i} \right\}.$$

Theorem 3.25. *Suppose that T is a compact operator on \mathcal{H} with $[T, J] = 0$ and that $p \in [1, +\infty)$. Then T is in $S_p(J)$ if and only if*

$$\sum_{n \in \mathbb{N}} |\langle \sigma_n, T e_n \rangle|^p < +\infty$$

for all orthonormal sets $\{e_n\}$ and $\{\sigma_n\}$ in $\mathcal{H}_+^{J_i}$. If T is also positive then

$$\|T\|_p = \sup \left\{ \left[\sum_{n \in \mathbb{N}} |\langle \sigma_n, T e_n \rangle|^p \right]^{\frac{1}{p}} : \{e_n\} \text{ and } \{\sigma_n\} \text{ orthonormal sets in } \mathcal{H}_+^{J_i} \right\}.$$

Theorem 3.26. Suppose that T is a compact operator on \mathcal{H} with $[T, J] = 0$ and that $0 < p \leq 2$. Then, for any orthonormal basis $\{e_n\}$ of \mathcal{H}_+^{Ji} , we have

$$\|T\|_p^p \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\langle e_k, Te_n \rangle|^p.$$

Proposition 3.27. Suppose T is a positive, compact operator on \mathcal{H} and x is a unit vector in \mathcal{H} . Then

- $\langle x, T^p x \rangle \geq \langle x, Tx \rangle^p$ for all $p \in [1, +\infty)$.
- $\langle x, T^p x \rangle \leq \langle x, Tx \rangle^p$ for all $0 < p \leq 1$.

Proof. Let $Tx = \sum_{n \in \mathbb{N}} e_n \lambda_n \langle e_n, x \rangle$ be a spectral decomposition for the operator T . Then, for all $p > 0$ and $x \in \mathcal{H}$, we have by the spectral theorem that

$$T^p x = \sum_{n \in \mathbb{N}} e_n \lambda_n^p \langle e_n, x \rangle$$

and thus

$$\langle x, T^p x \rangle = \sum_{n \in \mathbb{N}} \lambda_n^p |\langle e_n, x \rangle|^2.$$

We also know that for every Hilbert basis $(\xi_n)_{n \in \mathbb{N}}$ of \mathcal{H} we have that

$$\|x\|^2 = \sum_{n \in \mathbb{N}} |\langle \xi_n, x \rangle|^2, \quad \forall x \in \mathcal{H}.$$

Let us first assume that $p \geq 1$ and let q be its conjugate index. Applying Hölder's inequality gives

$$\begin{aligned} \langle x, Tx \rangle &= \left(\sum_{n \in \mathbb{N}} \lambda_n |\langle e_n, x \rangle|^2 \right) \\ &= \left(\sum_{n \in \mathbb{N}} \lambda_n |\langle e_n, x \rangle|^{\frac{2}{p}} |\langle e_n, x \rangle|^{\frac{2}{q}} \right) \\ &\leq \left(\sum_{n \in \mathbb{N}} \lambda_n^p |\langle e_n, x \rangle|^2 \right)^{\frac{1}{p}} \left(\sum_{n \in \mathbb{N}} |\langle e_n, x \rangle|^2 \right)^{\frac{1}{q}} \\ &= \langle x, T^p x \rangle^{\frac{1}{p}} \end{aligned}$$

and since $p \geq 1$ taking the p -th power preserves the inequality.

If $0 < p \leq 1$ then we can find $q \geq 1$ such that: $p + \frac{1}{q} = 1$. Using the Hölder inequality with the conjugate pair $(\frac{1}{p}, q)$ gives us that

$$\begin{aligned} \langle x, T^p x \rangle &= \sum_{n \in \mathbb{N}} \lambda_n^p |\langle e_n, x \rangle|^2 \\ &= \sum_{n \in \mathbb{N}} \lambda_n^p |\langle e_n, x \rangle|^{2p} |\langle e_n, x \rangle|^{\frac{2}{q}} \\ &\leq \left(\sum_{n \in \mathbb{N}} \lambda_n |\langle e_n, x \rangle|^2 \right)^p \left(\sum_{n \in \mathbb{N}} |\langle e_n, x \rangle|^2 \right)^{\frac{1}{q}} \\ &= \langle x, Tx \rangle^p \end{aligned}$$

and this finishes the proof. □

Corollary 3.28. *Suppose that T is a positive, compact operator on \mathcal{H} such that $[T, J] = 0$ and that $(e_n)_{n \in \mathbb{N}}$ is an orthonormal basis of \mathcal{H} . If $p \in [1, +\infty)$, then the condition*

$$\sum_{n \in \mathbb{N}} \langle e_n, T e_n \rangle^p < +\infty$$

is necessary for $T \in S_p(J)$. If $0 < p \leq 1$ then this condition is sufficient.

Proof. From Lemma 3.22 and Lemma 3.16, we have

$$T \in S_p(J) \iff T^p \in S_1(J) \iff \sum_{n \in \mathbb{N}} \langle e_n, T^p e_n \rangle < +\infty.$$

Applying Proposition 3.27 gives the result. □

Theorem 3.29. *Suppose T is a compact operator on \mathcal{H} with $[T, J] = 0$ and $p \geq 2$, then*

$$T \in S_p(J) \iff \sum_{n \in \mathbb{N}} \|T e_n\|^p < +\infty$$

for all orthonormal sets $\{e_n\}$ in $\mathcal{H}_+^{J_i}$. Moreover,

$$\|T\|_p = \sup \left\{ \left[\sum_{n \in \mathbb{N}} \|T e_n\|^p \right]^{\frac{1}{p}} \mid \{e_n\} \text{ orthonormal in } \mathcal{H}_+^{J_i} \right\}.$$

Proof. Since Res_{J_i} is p -norm preserving, this follows immediately from the corresponding results for complex linear operators [38, Theorem 1.33]. □

4 The Berezin Transform

Let \mathcal{H} be a quaternionic reproducing kernel Hilbert space of functions on the unit ball, that is a Hilbert space of left slice hyperholomorphic functions in the unit ball $\mathbb{D} \subset \mathbb{H}$ with the property that for each $w \in \mathbb{D}$ the point evaluation $f \mapsto f(w)$ is a bounded right linear functional on \mathcal{H} . From the Riesz representation theorem we know that there exists a unique function $K_w \in \mathcal{H}$ such that

$$f(w) = \langle K_w, f \rangle, \quad \forall f \in \mathcal{H}.$$

The function

$$K(q, w) := K_w(q), \quad q, w \in \mathbb{D}$$

is called the reproducing kernel of \mathcal{H} .

Lemma 4.1. *The functions $\{K_w \mid w \in \mathbb{D}\}$ span the entire space \mathcal{H} .*

Proof. This follows immediately from the reproducing property. If $f \perp K_w$ for all $w \in \mathbb{D}$, then

$$f(w) = \langle K_w, f \rangle = 0$$

and thus $f = 0$. □

Since we consider slice hyperholomorphic functions, we can use their specific structure to prove a stronger result.

Lemma 4.2. *Let $i \in \mathbb{S}$ and set $\mathbb{D}_i := \mathbb{D} \cap \mathbb{C}_i$. The set $\{K_w : w \in \mathbb{D}_i\}$ spans the space \mathcal{H} .*

Proof. If $f \perp K_w$ for any $w \in \mathbb{D}_i$, then $f(w) = \langle K_w, f \rangle = 0$ for all $w \in \mathbb{D}_i$. The representation formula, Theorem 2.15, then implies $f \equiv 0$, and hence $\mathcal{H} = \text{span}\{K_w : w \in \mathbb{D}_i\}$. \square

The kernel $K(q, w)$ is obviously left slice hyperholomorphic in q . Furthermore, it is right slice hyperholomorphic in \bar{w} , i.e. the mapping $w \mapsto K(q, \bar{w})$ is right slice hyperholomorphic. This follows from Corollary 2.7 because

$$\begin{aligned} K(q, \bar{w}) \bar{\partial}_{i\bar{w}}^\leftarrow &= \frac{1}{2} \langle K_q, K_{\bar{w}} \rangle (\partial_{w_0} + i_w \partial_{w_1}) \\ &= \frac{1}{2} \overline{(\partial_{w_0} - i_w \partial_{w_1}) \langle K_{\bar{w}}, K_q \rangle} \\ &= \frac{1}{2} \overline{(\partial_{w_0} + i_{\bar{w}} \partial_{w_1}) K_q(\bar{w})} = 0. \end{aligned}$$

Pointing out that

$$\|K_q\|^2 = K(q, q) \quad \text{and} \quad |K(q, w)|^2 \leq K(q, q)K(w, w),$$

we can conclude that, if for each $q \in \mathbb{D}$ there exists $f \in \mathcal{H}$ such that $f(q) \neq 0$, then

$$K(q, q) > 0 \quad \forall q \in \mathbb{D}.$$

We shall assume this to be true in the following. We can then normalize the reproducing kernels to obtain a family of unit vectors k_q by

$$k_q(w) = \frac{K(w, q)}{\sqrt{K(q, q)}} \quad \text{for } w \in \mathbb{D}.$$

We call these the normalized reproducing kernels of \mathcal{H} .

For the following discussion, we fix an imaginary unit $i \in \mathbb{S}$. Furthermore, we assume that there exists a unitary and antiselfadjoint operator J such that $\mathcal{H}_+^{Ji} = \text{span}_{\mathbb{C}_i}\{K_q, q \in \mathbb{D}_i\}$.

Definition 4.3. Let T be a bounded linear operator on \mathcal{H} such that $[T, J] = 0$. The function

$$\tilde{T}(q) = \langle k_q, T k_q \rangle, \quad q \in \mathbb{D}_i$$

is called the Berezin transform of T .

Proposition 4.4. *The Berezin transform has the following properties:*

- (i) *If T is self-adjoint, then \tilde{T} is real-valued.*
- (ii) *If T is positive, then \tilde{T} is non-negative.*
- (iii) *We have $\widetilde{T^*} = \overline{\tilde{T}}$.*
- (iv) *The mapping $T \mapsto \tilde{T}$ is a contractive \mathbb{C}_i -linear mapping from $\mathcal{B}_J(\mathcal{H})$ into $L^\infty(\mathbb{D}_i, \mathbb{H})$.*

Proof. For $q \in \mathbb{D}_i$, it is

$$\widetilde{T^*}(q) = \langle k_q, T^* k_q \rangle = \langle T k_q, k_q \rangle = \overline{\langle k_q, T k_q \rangle} = \overline{\tilde{T}(q)}$$

and hence (iii) holds. If T is self-adjoint, this implies $\widetilde{T}(q) = \tilde{T}(q)$ and so $\tilde{T}(q)$ is real. For positive T , the definition of positivity immediately implies (ii).

The Berezin transform is obviously \mathbb{R} -linear. Since $T \in \mathcal{B}_J(\mathcal{H})$, it maps $k_q \in \mathcal{H}_+^{Ji}$ to an element in \mathcal{H}_+^{Ji} , and hence its Berezin transform \tilde{T} takes values in \mathbb{C}_i . For $iT = JT$, we have

$$\tilde{iT}(q) = \langle k_q, JT k_q \rangle = \langle k_q, T k_q i \rangle = \tilde{T}(q)i = i\tilde{T}(q).$$

Thus, the Berezin transform is even \mathbb{C}_i -linear.

Finally, we deduce from the Cauchy-Schwarz-inequality that

$$\tilde{T}(q) = \langle k_q, T k_q \rangle \leq \|k_q\| \|T k_q\| \leq \|T\| \|k_q\|^2 = \|T\|.$$

Hence, $\tilde{T} \in L^\infty(\mathbb{D})$ with $\|\tilde{T}\|_\infty \leq \|T\|$.

□

Observe that the Berezin Transform of $T \in \mathcal{B}_J(\mathcal{H})$ coincides with the classical Berezin transform of the operator $\text{Res}_{Ji}(T) \in \mathcal{B}_J(\mathcal{H})$. Since the restriction operator Res_{Ji} and the classical Berezin transform are injective (cf. [38, Proposition 6.2.]), we immediately obtain the following Lemma.

Lemma 4.5. *The Berezin-transform is one-to-one.*

5 The Case of Weighted Bergman Spaces

In this section we consider the special case of weighted Bergman spaces on the unit ball. A first study of these spaces in the slice hyperholomorphic setting has been done in [14]. We recall the main definitions and results.

Definition 5.1. Let $i \in \mathbb{S}$ and let dm_i be the Lebesgue measure on the complex plane \mathbb{C}_i . For $\alpha > -1$, we define the measure $dA_{\alpha,i}(z)$ on the unit ball $\mathbb{D}_i := \mathbb{D} \cap \mathbb{C}_i$ in \mathbb{C}_i by

$$dA_{\alpha,i}(z) = \frac{\alpha+1}{\pi} (1-|z|^2)^\alpha dm_i(z).$$

For $p > 0$, the weighted slice Bergman space $\mathcal{A}_{\alpha,i}^p(\mathbb{D})$ is the quaternionic right vector space of all left slice hyperholomorphic functions f on \mathbb{D} such that

$$\int_{\mathbb{D}_i} |f(z)|^p dA_{\alpha,i}(z) < +\infty.$$

For $f \in \mathcal{A}_{\alpha,i}^p(\mathbb{D})$, we define

$$\|f\|_{p,\alpha,i} := \left(\int_{\mathbb{D}_i} |f(z)|^p dA_{\alpha,i}(z) \right)^{\frac{1}{p}}.$$

Corollary 5.2. *Let $i, j \in \mathbb{S}$ with $i \perp j$, let $f \in \mathcal{SH}_L(\mathbb{D})$ and write $f_i = f_1 + f_2 j$ with holomorphic functions $f_1, f_2 : \mathbb{D}_i \rightarrow \mathbb{C}_i$, cf. Lemma 2.8. Then $f \in \mathcal{A}_{\alpha,i}^p(\mathbb{D})$ if and only if f_1 and f_2 belong to the complex Bergman space $\mathcal{A}_{\mathbb{C},\alpha}^p(\mathbb{D})$, i.e. the space of all holomorphic functions g on \mathbb{D}_i such that $\int_{\mathbb{D}_i} |g(z)|^p dA_{\alpha,i}(z) < +\infty$.*

Lemma 5.3. *Let $\alpha > -1$, $p > 0$ and $i, j \in \mathbb{S}$. A left slice hyperholomorphic function f belongs to $\mathcal{A}_{\alpha,i}^p(\mathbb{D})$ if and only if it belongs to $\mathcal{A}_{\alpha,j}^p(\mathbb{D})$. Moreover,*

$$\|f\|_{p,\alpha,i}^p \leq 2^{\max\{p,1\}} \|f\|_{p,\alpha,j}^p \leq 2^{2\max\{p,1\}} \|f\|_{p,\alpha,i}^p.$$

Definition 5.4. For $\alpha > -1$ and $p > 0$, we define the weighted slice hyperholomorphic Bergman space $\mathcal{A}_\alpha^p(\mathbb{D})$ as the space of all left slice hyperholomorphic functions f such that

$$\|f\|_{p,\alpha} := \sup_{i \in \mathbb{S}} \|f\|_{p,\alpha,i} < +\infty.$$

Remark 5.5. Observe that Corollary 5.2 implies that, for each $i \in \mathbb{S}$, the spaces $\mathcal{A}_\alpha^p(\mathbb{D})$ and $\mathcal{A}_{\alpha,i}^p(\mathbb{D})$ contain the same elements and that their norms are equivalent.

5.1 Further properties of the weighted slice Bergman space $\mathcal{A}_{\alpha,i}^2(\mathbb{D})$

As in the complex space, the norm on the slice Bergman space $\mathcal{A}_{\alpha,i}^2(\mathbb{D})$ is generated by the scalar product

$$\langle f, g \rangle_{2,\alpha,i} := \int_{\mathbb{D}_i} \overline{f(z)} g(z) dA_{\alpha,i}(z). \quad (15)$$

This is however not true for the slice hyperholomorphic Bergman space $\mathcal{A}_{\alpha}^2(\mathbb{D})$: orthogonality and other concepts related to the scalar product will always depend on the complex plane chosen to define the scalar product. Nevertheless, by Lemma 5.3, independently of the choice of the complex plane, the norm topology on $\mathcal{A}_{\alpha}^2(\mathbb{D})$ is generated by the chosen scalar product.

Lemma 5.6. *Endowed with the scalar product defined in (15), the slice Bergman space $\mathcal{A}_{\alpha,i}^2(\mathbb{D})$ turns into a reproducing kernel quaternionic Hilbert space.*

Proof. If $f \in \mathcal{A}_{\alpha,i}^2(\mathbb{D})$ and $w \in \mathbb{D}$, then we can choose $j \in \mathbb{S}$ with $j \perp i_w$ and write the restriction of f to the plane \mathbb{C}_{i_w} as $f_{i_w} = f_1 + f_2 j$ with holomorphic functions $f_1, f_2 : \mathbb{D}_{i_w} \rightarrow \mathbb{C}_{i_w}$. By Corollary 5.2, the functions f_1 and f_2 belong to the complex Bergman space $\mathcal{A}_{\mathbb{C},\alpha}^2(\mathbb{D}_i)$. Since point evaluations are continuous functionals on $\mathcal{A}_{\mathbb{C},\alpha}^2(\mathbb{D}_i)$, cf. [38, Theorem 4.14], we have

$$\begin{aligned} |f(w)| &\leq |f_1(w)| + |f_2(w)| \\ &\leq C(\|f_1\|_{\mathbb{C},2,\alpha} + \|f_2\|_{\mathbb{C},2,\alpha}) \\ &\leq 2C\|f\|_{2,\alpha,i_w} \leq \tilde{C}\|f\|_{2,\alpha,i}, \end{aligned}$$

where the last equality follows from the equivalence of the Bergman slice norms, cf. Lemma 5.3. Hence, point evaluations are continuous linear functionals on $\mathcal{A}_{\alpha,i}^2(\mathbb{D})$. \square

Definition 5.7. Let $\alpha > -1$. We define the slice hyperholomorphic α -Bergman kernel for $q, w \in \mathbb{D}$ as

$$K_{\alpha}(q, w) := \frac{1}{2}(1 - i_q i) \frac{1}{(1 - q_{i_w} \overline{w})^{2+\alpha}} + \frac{1}{2}(1 + i_q i) \frac{1}{(1 - \overline{q_{i_w}} w)^{2+\alpha}}.$$

Remark 5.8. Observe that $K_{\alpha}(q, w)$ is an extension of the complex Bergman kernel

$$K_{\mathbb{C},\alpha}(z, w) = \frac{1}{(1 - z \overline{w})^{2+\alpha}}.$$

Whenever q and w belong to the same complex plane, $K_{\alpha}(q, w) = K_{\mathbb{C},\alpha}(q, w)$. Moreover, by a direct computation it is easy to see that the kernel $K_{\alpha}(x, w)$ is left slice hyperholomorphic in q and right slice hyperholomorphic in \overline{w} .

Lemma 5.9. *The function $K_{\alpha}(\cdot, \cdot)$ is the reproducing kernel of $\mathcal{A}_{\alpha,i}^2(\mathbb{D})$.*

Proof. Let $f \in \mathcal{A}_{\alpha,i}^2(\mathbb{D})$ and set $K_w(q) := K_{\alpha}(q, w)$ for $w \in \mathbb{D}$. We show that $K_w \in \mathcal{A}_{\alpha,i}^2(\mathbb{D})$ and $\langle K_w, f \rangle_{2,\alpha,i} = f(w)$.

First consider $w \in \mathbb{D}_i$. Since $K_w|_{\mathbb{D}_i}$ is nothing but the complex Bergman kernel $K_{\mathbb{C},\alpha}(\cdot, \cdot)$, we immediately obtain from Corollary 5.2 that K_w belongs to $\mathcal{A}_{\alpha,i}^2(\mathbb{D})$ and that we can write $f_i = f_1 + f_2 j$ with $f_1, f_2 \in \mathcal{A}_{\mathbb{C},\alpha}^2(\mathbb{D}_i)$. From Remark 5.8 and the reproducing property of $K_{\mathbb{C},\alpha}(\cdot, \cdot)$ in $\mathcal{A}_{\mathbb{C},\alpha}^2(\mathbb{D})$, we obtain

$$\begin{aligned} \langle K_w, f \rangle_{2,\alpha,i} &= \int_{\mathbb{D}_i} \overline{K_{\alpha}(z, w)} f(z) dA_{\alpha,i}(z) \\ &= \int_{\mathbb{D}_i} \overline{K_{\mathbb{C},\alpha}(z, w)} f_1(z) dA_{\alpha,i}(z) + \int_{\mathbb{D}_i} \overline{K_{\mathbb{C},\alpha}(z, w)} f_2(z) dA_{\alpha,i}(z) j \\ &= f_1(w) + f_2(w) j = f(w). \end{aligned}$$

If $w \notin \mathbb{D}_i$, then Theorem 2.15 implies

$$K_w = K_{w_i}(1 - ii_w)\frac{1}{2} + K_{\overline{w_i}}(1 + ii_w)\frac{1}{2},$$

because of the right slice hyperholomorphicity of $K_\alpha(q, w)$ in \overline{w} . Hence, the function K_w belongs to $\mathcal{A}_{\alpha,i}^2(\mathbb{D})$ because it is a right linear combination of the functions K_{w_i} and $K_{\overline{w_i}}$, which belong to $\mathcal{A}_{\alpha,i}^2(\mathbb{D})$ by the above argumentation. From the representation formula, we finally also deduce

$$\begin{aligned} \langle K_w, f \rangle_{\alpha,i} &= \frac{1}{2} \overline{(1 - ii_w)} \langle K_{w_i}, f \rangle_{\alpha,i} + \frac{1}{2} \overline{(1 + ii_w)} \langle K_{\overline{w_i}}, f \rangle_{\alpha,i} \\ &= \frac{1}{2} (1 - i_w i) f(w_i) + \frac{1}{2} (1 + i_w i) f(\overline{w_i}) = f(w). \end{aligned}$$

□

5.2 The Berezin transform on $\mathcal{A}_{\alpha,i}^2(\mathbb{D})$

In this section we consider the following fixed unitary and anti-selfadjoint operator

$$Jf := \text{ext}(if_i), \quad \forall f \in \mathcal{A}_{\alpha,i}^2(\mathbb{D})$$

and consider it as a left scalar multiplication with $i \in \mathbb{S}$.

Corollary 5.10. *It is*

$$(\mathcal{A}_{\alpha,i}^2(\mathbb{D}))_+^{J_i} = \overline{\text{span}_{\mathbb{C}_i}(K_q : q \in \mathbb{D}_i)} \cong \mathcal{A}_{\mathbb{C},\alpha}^2(\mathbb{D}_i).$$

Proof. Write f_i for $f \in \mathcal{A}_{\alpha,i}^2(\mathbb{D})$ as $f_i = f_1 + f_2 j$ with components that are holomorphic on \mathbb{D}_i . By the right linearity of the extension operator, we then have $f = \text{ext}(f_1) + \text{ext}(f_2)j$. From

$$\begin{aligned} Jf &= \text{ext}(if_i) \\ &= \text{ext}(f_1 i) + \text{ext}(f_2 i j) \\ &= \text{ext}(f_1) i - \text{ext}(f_2) j i, \end{aligned}$$

we deduce that $f \in (\mathcal{A}_{\alpha,i}^2(\mathbb{D}))_+^{J_i}$ if and only if $f = \text{ext}(f_1)$, i.e. if and only if $f_2 = 0$. The mapping $\varphi : f \mapsto f_i$ is therefore an isometric isomorphism between $(\mathcal{A}_{\alpha,i}^2(\mathbb{D}))_+^{J_i}$ and $\mathcal{A}_{\mathbb{C},\alpha}^2(\mathbb{D}_i)$ and so $(\mathcal{A}_{\alpha,i}^2(\mathbb{D}))_+^{J_i} \cong \mathcal{A}_{\mathbb{C},\alpha}^2(\mathbb{D}_i)$. Since $\varphi(K_q) = K_{\mathbb{C},q}$ and $\text{span}_{\mathbb{C}_i}\{K_{\mathbb{C},q}, q \in \mathbb{D}_i\}$ is dense in $\mathcal{A}_{\mathbb{C},\alpha}^2(\mathbb{D}_i)$, we obtain that

$$\begin{aligned} \mathcal{A}_{\alpha,i}^2(\mathbb{D}) &= \varphi^{-1}(\mathcal{A}_{\mathbb{C},\alpha}^2(\mathbb{D}_i)) \\ &= \varphi^{-1}(\overline{\text{span}_{\mathbb{C}_i}\{K_{\mathbb{C},q}, q \in \mathbb{D}_i\}}) \\ &= \overline{\text{span}_{\mathbb{C}_i}\{\varphi^{-1}(K_{\mathbb{C},q}), q \in \mathbb{D}_i\}} \\ &= \overline{\text{span}_{\mathbb{C}_i}\{K_q, q \in \mathbb{D}_i\}}. \end{aligned}$$

□

Recall that a bounded linear operator T on $\mathcal{A}_{\alpha,i}^2(\mathbb{D})$ satisfies $[T, J] = 0$ if and only if $T = \text{Lift}_{J_i}(T_{\mathbb{C}})$ for some bounded linear operator on $T_{\mathbb{C}}$ on the complex Bergman space $\mathcal{A}_{\mathbb{C},\alpha}^2(\mathbb{D}_i)$. In this case $\widetilde{T}(z) = \widetilde{T_{\mathbb{C}}}(z)$, where $\widetilde{T_{\mathbb{C}}}$ denotes the Berezin transform of the complex operator $T_{\mathbb{C}}$.

Theorem 5.11. *Let T be a bounded positive operator on $\mathcal{A}_{\alpha,i}^2(\mathbb{D})$ and let $(\lambda_n)_{n \in \mathbb{N}}$ be its sequence of singular values. If*

$$d\mu_i(z) = \frac{1}{\pi(1 - |z|^2)^2} dm_i$$

is the \mathbb{C}_i -Möbius invariant area measure on \mathbb{D}_i , then

$$\sum_{n=1}^{+\infty} \lambda_n = (\alpha + 1) \int_{\mathbb{D}_i} \tilde{T}(z) d\mu_i(z).$$

Proof. Let $(e_n)_{n \in \mathbb{N}}$ be any orthonormal basis of $\mathcal{A}_{\alpha,i}^2(\mathbb{D})$. By Lemma 3.16, we have $\sum_{n=1}^{+\infty} \lambda_n = \sum_{n=0}^{+\infty} \langle e_n, T e_n \rangle$. Set $S = \sqrt{T}$. Fubini's theorem, the reproducing property of K_z and Parseval's identity imply

$$\begin{aligned} \sum_{n=1}^{+\infty} \langle e_n, T e_n \rangle &= \sum_{n=1}^{+\infty} \|S e_n\|^2 = \sum_{n=1}^{+\infty} \int_{\mathbb{D}_i} |S e_n(z)|^2 dA_{\alpha,i}(z) \\ &= \int_{\mathbb{D}_i} \sum_{n=1}^{+\infty} |S e_n(z)|^2 dA_{\alpha,i}(z) = \int_{\mathbb{D}_i} \sum_{n=1}^{+\infty} |\langle K_z, S e_n \rangle|^2 dA_{\alpha,i}(z) \\ &= \int_{\mathbb{D}_i} \sum_{n=1}^{+\infty} |\langle e_n, S K_z \rangle|^2 dA_{\alpha,i}(z) = \int_{\mathbb{D}_i} \|S K_z\|^2 dA_{\alpha,i}(z) \\ &= \int_{\mathbb{D}_i} \langle K_z, T K_z \rangle dA_{\alpha,i}(z) = \int_{\mathbb{D}_i} \tilde{T}(z) K(z, z) dA_{\alpha,i}(z) \\ &= (\alpha + 1) \int_{\mathbb{D}_i} \tilde{T}(z) d\mu_i(z). \end{aligned}$$

□

We have the following important consequence:

Corollary 5.12. *Let J be any unitary anti-selfadjoint operator on $\mathcal{A}_{\alpha,i}^2(\mathbb{D})$. A positive operator $T \in \mathcal{B}_J(\mathcal{A}_{\alpha,i}^2(\mathbb{D}))$ belongs to the trace class $S_1(J)$ if and only if $\tilde{T} \in L^1(\mathbb{D}_i, d\mu_i)$.*

Corollary 5.13. *If $T \in S_1(J)$, then \tilde{T} is in $L^1(\mathbb{D}_i, d\mu_i)$ and*

$$\mathrm{Tr}_{Ji}(T) = (\alpha + 1) \int_{\mathbb{D}_i} \tilde{T}(z) d\mu_i(z).$$

Proof. Set $T_{\mathbb{C}} = \mathrm{Res}_{Ji}(T)$, write

$$T_{\mathbb{C}} = T_{\mathbb{C},1} - T_{\mathbb{C},2} + i(T_{\mathbb{C},2} - T_{\mathbb{C},3})$$

with positive operators $T_{\mathbb{C},\ell} \in \mathcal{B}(\mathcal{A}_{\mathbb{C},\alpha}^2(\mathbb{D}_i))$ and set $T_\ell := \mathrm{Lift}_{Ji}(T_{\mathbb{C},\ell})$ for $\ell = 1, \dots, 4$. Then the operators T_ℓ are positive and $T = T_1 - T_2 + JT_3 - JT_4$. By the \mathbb{C}_i -linearity of the Ji -trace, the \mathbb{C}_i -linearity of the Berezin transform and Theorem 5.11, we have

$$\begin{aligned} \mathrm{Tr}_{Ji}(T) &= \mathrm{Tr}_{Ji}(T_1) - \mathrm{Tr}_{Ji}(T_2) + i \mathrm{Tr}_{Ji}(T_3) - i \mathrm{Tr}_{Ji}(T_4) \\ &= (\alpha + 1) \int_{\mathbb{D}_i} \widetilde{T}_1(z) d\mu_i(z) - (\alpha + 1) \int_{\mathbb{D}_i} \widetilde{T}_2(z) d\mu_i(z) \\ &\quad + i(\alpha + 1) \int_{\mathbb{D}_i} \widetilde{T}_3(z) d\mu_i(z) - i(\alpha + 1) \int_{\mathbb{D}_i} \widetilde{T}_4(z) d\mu_i(z) \\ &= (\alpha + 1) \int_{\mathbb{D}_i} \tilde{T}(z) d\mu_i(z). \end{aligned}$$

□

Theorem 5.14. *Let T be a positive operator on $\mathcal{A}_{\alpha,i}^2(\mathbb{D})$ such that $[T, J] = 0$.*

(i) *If $1 \leq p < +\infty$ and $T \in S_p(J)$, then $\tilde{T} \in L^p(\mathbb{D}_i, d\mu_i)$.*

(ii) If $0 < p \leq 1$ and $\tilde{T} \in L^p(\mathbb{D}_i, d\mu_i)$, then $T \in S_p(J)$.

Proof. For $1 \leq p < +\infty$, Proposition 3.27 implies $\tilde{T}^p(z) \geq (\tilde{T}(z))^p$ for $z \in \mathbb{D}_i$. Because $T \in S_p(J)$, we have by Lemma 3.22 that $T^p \in S_1(J)$, and hence, we deduce from Theorem 5.11

$$\int_{\mathbb{D}_i} \left(\tilde{T}(z) \right)^p d\mu_i(z) \leq \int_{\mathbb{D}_i} \tilde{T}^p(z) d\mu_i(z) = \text{Tr}_{Ji}(T^p) < +\infty.$$

The second statement is proved in a similar way. □

Corollary 5.15. *If $1 \leq p < +\infty$ and $T \in S_p(J)$, then $\tilde{T} \in L^p(\mathbb{D}_i, d\mu_i)$.*

Proof. This follows from Theorem 5.14 and the fact that we can write T , in terms of the \mathbb{C}_i -Banach space structure defined on $S_p(J)$, as a \mathbb{C}_i -linear combination of positive operators. See the proof of Corollary 5.13. □

For $z \in \mathbb{D}_i$, we denote by P_z the orthogonal projection of $\mathcal{A}_{\alpha,i}^2(\mathbb{D})$ onto the one-dimensional subspace generated by k_z , i.e.

$$P_z(f) = k_z \langle k_z, f \rangle, \quad \forall f \in \mathcal{A}_{\alpha,i}^2(\mathbb{D}).$$

Obviously, P_z is a positive operator with rank one. Observe that $k_z \in \mathcal{H}_+^{Ji}$ by Corollary 5.10 and hence $Jk_z = k_z i$, from which we deduce

$$\begin{aligned} JP_z f &= Jk_z \langle k_z, f \rangle = k_z i \langle k_z, f \rangle \\ &= k_z \langle -k_z i, f \rangle = k_z \langle J^* k_z, f \rangle \\ &= k_z \langle k_z, Jf \rangle \\ &= P_z Jf \end{aligned}$$

for all $f \in \mathcal{A}_{\alpha,i}^2(\mathbb{D})$. Hence, $[P_z, J] = 0$ and $P_z \in S_1(J)$ as $\text{Tr}_{Ji}(P_z) = 1$.

Corollary 5.16. *Let $T \in \mathcal{B}_J(\mathcal{A}_{\alpha,i}^2(\mathbb{D}))$. Then*

$$\tilde{T}(z) = \text{Tr}_{Ji}(TP_z), \quad \forall z \in \mathbb{D}_i.$$

Proof. Let $(e_n)_{n \in \mathbb{N}}$ be an orthonormal basis of $(\mathcal{A}_{\alpha,i}^2(\mathbb{D}))_+^{Ji}$ with $e_1 = k_z$. Then

$$\text{Tr}_{Ji}(TP_z) = \sum_{n=1}^{+\infty} \langle e_n, TP_z e_n \rangle = \langle k_z, Tk_z \rangle = \tilde{T}(z).$$

□

Corollary 5.17. *Let $T \in \mathcal{B}_J(\mathcal{A}_{\alpha,i}^2(\mathbb{D}))$. Then*

$$|\tilde{T}(z) - \tilde{T}(w)| \leq \|T\| \|P_z - P_w\|_1, \quad \forall z \in \mathbb{D}_z.$$

Proof. By Corollary 5.16, we have

$$\tilde{T}(z) - \tilde{T}(w) = \text{Tr}_{Ji}(TP_z) - \text{Tr}_{Ji}(TP_w) = \text{Tr}_{Ji}(T(P_z - P_w)).$$

Hence, we deduce from Corollary 3.13 and (iii) in Lemma 3.19 that

$$|\tilde{T}(z) - \tilde{T}(w)| \leq |\text{Tr}_{Ji}(T(P_z - P_w))| \leq \|T\| \|P_z - P_w\|_1.$$

□

Lemma 5.18. *Let $z, w \in \mathbb{D}_i$. Then*

$$\|P_z - P_w\| = (1 - |\langle k_w, k_z \rangle|^2)^{\frac{1}{2}},$$

while

$$\|P_z - P_w\|_1 = 2(1 - |\langle k_w, k_z \rangle|^2)^{\frac{1}{2}}.$$

Proof. These equalities follow immediately from the corresponding equalities in the complex case, cf. [38, Lemma 6.10], and the following facts: $\text{Res}_{J,i}$ preserves both the operator norm and the Schatten norm (cf. Remark 2.32 and Corollary 3.12) and $\langle k_w, k_z \rangle = \langle k_{w,i}, k_{z,i} \rangle_{\mathbb{C}}$, where $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ denotes the scalar product of the complex Bergman space $\mathcal{A}_{\mathbb{C},i}^2(\mathbb{D}_i)$ and $k_{w,i}$ denotes the restriction of k_w to \mathbb{D}_i . Indeed, denoting $P_{z,\mathbb{C}} := \text{Res}_{J,i}(P_z)$ and $P_{w,\mathbb{C}} := \text{Res}_{J,i}(P_w)$, we have

$$\|P_z - P_w\| = \|P_{z,\mathbb{C}} - P_{w,\mathbb{C}}\| = (1 - |\langle k_{w,i}, k_{z,i} \rangle_{\mathbb{C}}|^2)^{\frac{1}{2}} = (1 - |\langle k_w, k_z \rangle|^2)^{\frac{1}{2}}.$$

The case of the Schatten-norm follows analogously. □

Finally, we obtain Lipschitz estimates for the Berezin transform analogue to those of the complex case.

Theorem 5.19. *Let $T \in \mathcal{B}_J(\mathcal{A}_{\alpha,i}^2(\mathbb{D}))$. Then*

$$|\tilde{T}(z) - \tilde{T}(w)| \leq 2\sqrt{2+\alpha}\|T\|\rho(z, w), \quad \forall z, w \in \mathbb{D}_i,$$

where

$$\rho(z, w) = \frac{|z - w|}{|1 - z\overline{w}|}$$

is the pseudo-hyperbolic metric between z and w . Furthermore, the Lipschitz constant $2\sqrt{2+\alpha}$ is sharp.

Proof. Again, we apply the corresponding result for complex linear operators [38, Theorem 6.11]. Denoting $T_{\mathbb{C}} = \text{Res}_{J,i}(T)$, we have

$$\begin{aligned} |\tilde{T}(z) - \tilde{T}(w)| &= |\widetilde{T_{\mathbb{C}}}(z) - \widetilde{T_{\mathbb{C}}}(w)| \\ &\leq 2\sqrt{2+\alpha}\|T_{\mathbb{C}}\|\rho(z, w) \\ &= 2\sqrt{2+\alpha}\|T\|\rho(z, w). \end{aligned}$$
□

Theorem 5.20. *Let $T \in \mathcal{B}_J(\mathcal{A}_{\alpha,i}^2(\mathbb{D}))$. Then*

$$|\tilde{T}(z) - \tilde{T}(w)| \leq 2\sqrt{2+\alpha}\|T\|\beta(z, w), \quad \forall z, w \in \mathbb{D}_i,$$

where

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)},$$

is the Bergman metric on \mathbb{D}_i . Furthermore, the Lipschitz constant $2\sqrt{2+\alpha}$ is sharp.

Proof. Once more, we deduce this from the corresponding result for complex linear operators [38, Theorem 6.11]. Denoting $T_{\mathbb{C}} = \text{Res}_{J,i}(T)$, we have

$$\begin{aligned} |\tilde{T}(z) - \tilde{T}(w)| &= |\widetilde{T_{\mathbb{C}}}(z) - \widetilde{T_{\mathbb{C}}}(w)| \\ &\leq 2\sqrt{2+\alpha}\|T_{\mathbb{C}}\|\beta(z, w) \\ &= 2\sqrt{2+\alpha}\|T\|\beta(z, w). \end{aligned}$$

and this concludes the proof. □

We conclude this paper with a remark on further applications of slice hyperholomorphicity and quaternionic operators.

Remark 5.21. Classical Schur analysis is an important branch of operators theory with several applications in science and in technology, see for example the book [2], the notion of S -spectrum and of S -resolvent operators appear in Schur analysis in the slice hyperholomorphic setting in the realization of Schur functions, see the foundational paper [7].

The literature on Schur analysis in the slice hyperholomorphic setting is nowadays very well developed we mention just some of the main results that are contained in the papers [7–9, 11] and in the book [6]. The main reference for slice hyperholomorphic functions are the books [6, 20, 27].

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